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- White Paper -

Space-Time Block-Codes (STBCs)

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Document Version 1.0
Darmstadt, 9th February 2024

Keywords:
Transmit Diversity, Cooperative Communication, Virtual Transmit Cluster

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Notation

Variables

a	Scalar
$\mathbf{a}$	Vector
$\mathbf{A}$	Matrix

Super-Scripts

$(\cdot)^T$	Transpose
$(\cdot)^*$	Complex Conjugate
$(\cdot)^H$	Adjoint (Transpose Complex Conjugate)

1 Introduction

Space-Time Block Codes (STBCs) are in general designed to achieve a transmit diversity gain and have been first proposed in 1998 by Siavash Alamouti [1].

Diversity is typically used for wireless communication systems to mitigate fading. Thereby, fading implies variations in the received signal power which occur due to multipath propagation, deflections, shadowing, etc. If a channel is in a *deep fade* the signal strength is very low which can lead to tremendous bit error rates.
Basically, diversity means that several copies of the same information (but not necessarily of the same transmit signal) are used. To achieve the best results, all links should be used to transmit one of these copies.
Considering the specific realization, different degrees of freedom can be utilized to benefit from a diversity gain, namely space, time and frequency. For spatial diversity several antennas are used. Thereby, the antennas can be all located on the same transmitter (*co-located setup*) or several transmitters each equipped with one single antenna are used (*distributed setup*). A *time diversity* can be achieved if the copies are spread over several time slots. In contrast, different sub-carriers on varying carrier frequencies are used for *frequency diversity*.

One very important reference paper summarizing the achievable diversity gain for MISO systems has been published by Tarokh et al. in 1998 [2]. With this research work, the authors could prove that the maximum achievable diversity gain in a MISO system is equal to the number of transmitters N_{TX}. Moreover, they deliver a rank and determinant criterion. While the rank criterion defines conditions for a space code to reach the maximal diversity gain, the determinant criterion defines those conditions which are necessary to reach a coding or respectively SNR gain.

A diversity gain becomes visible in a BER vs. SNR comparison. It increases the slope of the graph. The higher the diversity gain, the higher the slope's magnitude, whereby the magnitude is exactly equal to the diversity gain. Contrary, a SNR gain leads to a shift to the left on the x-axis.

Subsequently, some outstanding STBCs are presented with some detail. For the further description of the different coding schemes a representation in matrix form is quite convenient. Thereby, the rows of the matrix are assigned to the time or respectively time slots while the columns correspond to the different transmitting nodes or, respectively, antennas. A matrix representation for a coding scheme, where 3 symbols are transmitted in 4 time slots could look like:

$$\mathbf{C} = \begin{bmatrix} x_1 & x_2 & x_3 \\ -x_3 & x_1^* & -x_2 \\ x_1^* & -x_3^* & -x_2^* \\ x_3^* & x_2^* & -x_1 \end{bmatrix} \tag{1.1}$$

From the coding matrix it can be directly observed, that antenna 1 transmits x_1, $-x_3$, x_1^* and x_3^* in four consecutive time slots. Similarly, antenna 2 transmits x_2, x_1^*, $-x_3^*$ and x_2^* while antenna 3 sends x_3, $-x_2$, x_2^* and $-x_1$.

Furthermore, the matrix allows to determine the code rate R_C which is defined as the ratio between the number of different information symbols and the necessary time slots for the transmission. For the example above, it follows for R_C that

$$R_\mathrm{C} = \frac{3}{4} = 0.75 = 75\%. \tag{1.2}$$

STBCs can be classified with respect to the *channel state information (CSI)*. Thereby, it is distinguished between codes which require

- no CSI at all,
- CSI at the transmit side (CSIT) and
- CSI at the receiving side (CSIR).

From main interest are those codes which rely on CSIR. This is because, CSIT is very complex to achieve especially for large scale networks. In contrast, differential STBCs which do not need any CSI and that were first introduced by Tarokh and Jafarkhani in [3] involve a huge decoding effort.

All presented examples, expect for the Alamouti code, are based on a setup where four transmit signals ($N_\mathrm{TX} = 4$) are required. This could either be a single node equipped with four transmit antennas or equivalently four single-antenna nodes. Unfortunately, the Alamouti code is limited to $N_\mathrm{TX} = 2$.

2 Alamouti Code

According to [1] the encoding matrix of the Alamouti scheme is given as

$$\mathbf{X}_{\text{Alamouti}} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ -\alpha_2^* & \alpha_1^* \end{bmatrix}. \tag{2.1}$$

So, the physical signal on the channel $\mathbf{y}_{\text{ch}}$ assuming a quasi-static fading environment follows as

$$\mathbf{y}_{\text{ch}} = \mathbf{X}_{\text{Alamouti}} \cdot \mathbf{h} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ -\alpha_2^* & \alpha_1^* \end{bmatrix} \cdot \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 h_1 + \alpha_2 h_2 \\ -\alpha_2^* h_1 + \alpha_1^* h_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 h_1 + \alpha_2 h_2 \\ \alpha_1^* h_2 - \alpha_2^* h_1 \end{pmatrix}. \tag{2.2}$$

Using this expression, the equivalent channel matrix $\mathcal{H}$ can be denoted as

$$\mathcal{H} = \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix}, \tag{2.3}$$

so, that

$$\mathbf{y} = \mathcal{H} \cdot \boldsymbol{\alpha} = \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 h_1 + \alpha_2 h_2 \\ \alpha_1 h_2^* - \alpha_2 h_1^* \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_{\text{ch},1} \\ y_{\text{ch},2}^* \end{pmatrix}. \tag{2.4}$$

Decoding can be performed by a multiplication with $\mathcal{H}^H$. Hence,

$$\hat{\boldsymbol{\alpha}} = \mathcal{H}^H \cdot \mathbf{y} = \mathcal{H}^H \cdot \mathcal{H} \cdot \boldsymbol{\alpha}. \tag{2.5}$$

Assuming quasi-static channels the orthogonality is given, wherefore

$$\mathcal{H}^H \cdot \mathcal{H} = \begin{bmatrix} h_1^* & h_2 \\ h_2^* & -h_1 \end{bmatrix} \cdot \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix} = (|h_1|^2 + |h_2|^2) \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.6}$$

Thus, decoding can be also formulated as

$$\hat{\alpha_1} = h_1^* \cdot y_{\text{ch},1} + h_2 \cdot y_{\text{ch},2}^* \quad \text{and} \quad \hat{\alpha_2} = h_2^* \cdot y_{\text{ch},1} - h_1 \cdot y_{\text{ch},2}^*. \tag{2.7}$$

Inserting $y_{\text{ch},1}$ and $y_{\text{ch},2}^*$, these equations become

$$\begin{aligned} \hat{\alpha_1} &= h_1^* \cdot (\alpha_1 h_1 + \alpha_2 h_2) + h_2 \cdot (\alpha_1 h_2^* - \alpha_2 h_1^*) \\ \hat{\alpha_2} &= h_2^* \cdot (\alpha_1 h_1 + \alpha_2 h_2) - h_1 \cdot (\alpha_1 h_2^* - \alpha_2 h_1^*). \end{aligned} \tag{2.8}$$

Simplifying directly leads to

$$\begin{aligned} \hat{\alpha_1} &= \alpha_1 h_1 h_1^* + \cancel{\alpha_2 h_2 h_1^*} + \alpha_1 h_2^* h_2 - \cancel{\alpha_2 h_1^* h_2} = \alpha_1 \cdot (|h_1|^2 + |h_2|^2) \\ \hat{\alpha_2} &= \cancel{\alpha_1 h_1 h_2^*} + \alpha_2 h_2 h_2^* - \cancel{\alpha_1 h_2^* h_1} + \alpha_2 h_1^* h_1 = \alpha_2 \cdot (|h_1|^2 + |h_2|^2). \end{aligned} \tag{2.9}$$

The interference terms cancel and a sum of the channel coefficients' absolute values is left which is the typical behaviour for OSTBC in a quasi-static environment. However, this beneficial bearing is lost in a time-variant environment. If a different Rayleigh-fading channel vector is used for every time-slot, the signal on the physical channel has to be expressed as

$$\mathbf{y}_{\text{ch}} = \mathbf{X}_{\text{Alamouti}} \cdot \mathbf{h}(t) = \begin{pmatrix} \alpha_1 h_{11} + \alpha_2 h_{21} \\ \alpha_1^* h_{22} - \alpha_2^* h_{12} \end{pmatrix}. \tag{2.10}$$

Thus, the equivalent channel matrix $\mathcal{H}$ has to be adapted to

$$\mathcal{H} = \begin{bmatrix} h_{11} & h_{21} \\ h_{22}^* & -h_{12}^* \end{bmatrix} \tag{2.11}$$

The orthogonality is lost which becomes evident with respect to the decoding rule, i. e.

$$\begin{aligned} \mathcal{H}^H \cdot \mathcal{H} &= \begin{bmatrix} h_{11}^* & h_{22} \\ h_{21}^* & -h_{12} \end{bmatrix} \cdot \begin{bmatrix} h_{11} & h_{21} \\ h_{22}^* & -h_{12}^* \end{bmatrix} = \begin{bmatrix} h_{11} h_{11}^* + h_{22} h_{22}^* & h_{11}^* h_{21} - h_{12}^* h_{22} \\ h_{11} h_{21}^* - h_{12} h_{22}^* & h_{21} h_{21}^* + -h_{12} h_{12}^* \end{bmatrix} \\ &= \begin{bmatrix} |h_{11}|^2 + |h_{22}|^2 & h_{11}^* h_{21} - h_{12}^* h_{22} \\ h_{11} h_{21}^* - h_{12} h_{22}^* & |h_{21}|^2 + |h_{12}|^2 \end{bmatrix}. \end{aligned} \tag{2.12}$$

Inter-symbol-interference (ISI) is introduced, wherefore the decoding has to be adapted as well. A decoding by multiplying with $\mathcal{H}^H$ is no longer possible.

3 Orthogonal STBC (OSTBC)

With the exception of a few special cases, OSTBC can only achieve a code-rate $R_\mathrm{C} = \frac{1}{2}$ for four or more transmitters. Hence, to keep the generality, the encoding matrix can be selected as

$$\mathbf{X}_\mathrm{OSTBC} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ -\alpha_2 & \alpha_1 & -\alpha_4 & \alpha_3 \\ -\alpha_3 & \alpha_4 & \alpha_1 & -\alpha_2 \\ -\alpha_4 & -\alpha_3 & \alpha_2 & \alpha_1 \\ \alpha_1^* & \alpha_2^* & \alpha_3^* & \alpha_4^* \\ -\alpha_2^* & \alpha_1^* & -\alpha_4^* & \alpha_3^* \\ -\alpha_3^* & \alpha_4^* & \alpha_1^* & -\alpha_2^* \\ -\alpha_4^* & -\alpha_3^* & \alpha_2^* & \alpha_1^* \end{bmatrix}. \tag{3.1}$$

Assuming quasi-static fading channels, the physical signal on the transmit channel $\mathbf{y}_\mathrm{ch}$ can be obtained by a multiplication with the channel vector as

$$\mathbf{y}_\mathrm{ch} = \mathbf{X}_\mathrm{OSTBC} \cdot \mathbf{h} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ -\alpha_2 & \alpha_1 & -\alpha_4 & \alpha_3 \\ -\alpha_3 & \alpha_4 & \alpha_1 & -\alpha_2 \\ -\alpha_4 & -\alpha_3 & \alpha_2 & \alpha_1 \\ \alpha_1^* & \alpha_2^* & \alpha_3^* & \alpha_4^* \\ -\alpha_2^* & \alpha_1^* & -\alpha_4^* & \alpha_3^* \\ -\alpha_3^* & \alpha_4^* & \alpha_1^* & -\alpha_2^* \\ -\alpha_4^* & -\alpha_3^* & \alpha_2^* & \alpha_1^* \end{bmatrix} \cdot \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix} = \begin{pmatrix} \alpha_1 h_1 + \alpha_2 h_2 + \alpha_3 h_3 + \alpha_4 h_4 \\ \alpha_1 h_2 - \alpha_2 h_1 + \alpha_3 h_4 - \alpha_4 h_3 \\ \alpha_1 h_3 - \alpha_2 h_4 - \alpha_3 h_1 + \alpha_4 h_2 \\ \alpha_1 h_4 + \alpha_2 h_3 - \alpha_3 h_2 - \alpha_4 h_1 \\ \alpha_1^* h_1 + \alpha_2^* h_2 + \alpha_3^* h_3 + \alpha_4^* h_4 \\ \alpha_1^* h_2 - \alpha_2^* h_1 + \alpha_3^* h_4 - \alpha_4^* h_3 \\ \alpha_1^* h_3 - \alpha_2^* h_4 - \alpha_3^* h_1 + \alpha_4^* h_2 \\ \alpha_1^* h_4 + \alpha_2^* h_3 - \alpha_3^* h_2 - \alpha_4^* h_1 \end{pmatrix}. \tag{3.2}$$

Considering the derived expression for the physical signal on the channel, the transmission can also be modelled using an equivalent channel matrix $\mathcal{H}$. Accordingly,

$$\mathbf{y} = \mathcal{H} \cdot \boldsymbol{\alpha} = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 \\ h_2 & -h_1 & h_4 & -h_3 \\ h_3 & -h_4 & -h_1 & h_2 \\ h_4 & h_3 & -h_2 & -h_1 \\ h_1^* & h_2^* & h_3^* & h_4^* \\ h_2^* & -h_1^* & h_4^* & -h_3^* \\ h_3^* & -h_4^* & -h_1^* & h_2^* \\ h_4^* & h_3^* & -h_2^* & -h_1^* \end{bmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \end{pmatrix} = \begin{pmatrix} y_{\mathrm{ch},1} \\ y_{\mathrm{ch},2} \\ y_{\mathrm{ch},3} \\ y_{\mathrm{ch},4} \\ y_{\mathrm{ch},5}^* \\ y_{\mathrm{ch},6}^* \\ y_{\mathrm{ch},7}^* \\ y_{\mathrm{ch},8}^* \end{pmatrix}. \tag{3.3}$$

It is apparent, that the equivalent channel model corresponds to the actual signal if the complex conjugate value is used for the last half of the received samples. Using the equivalent channel model and exploiting the orthogonality of the encoding matrix and thus the orthogonality of $\mathcal{H}$, decoding can be performed by a multiplication with $\mathcal{H}^H$. Thus,

$$\hat{\boldsymbol{\alpha}} = \mathcal{H}^H \cdot \mathbf{y} = \mathcal{H}^H \cdot \mathcal{H} \cdot \boldsymbol{\alpha}. \tag{3.4}$$

For an exemplary setup with four transmitters which are used to transmit four information symbols, the matched multiplication than follows as

$$\mathcal{H}^H \cdot \mathcal{H} = \mathcal{H}_\mathrm{m} = 2 \cdot (|h_1|^2 + |h_2|^2 + |h_3|^2 + |h_4|^2) \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.5}$$

It can easily be seen, that the terms which would cause ISI cancel out. Moreover, it is worth denoting, that only the absolute values of the complex channel coefficients are used. Thus, the superimposing is always constructive and never destructive, which is the reason for the attainable transmit diversity gain.

If the transmit channels are assumed to be maximally time-variant instead of quasi-static so that a different channel vector is used for each time slot, the physical signal on the channel has to be expressed as

$$\mathbf{y}_\mathrm{ch} = \mathbf{X}_\mathrm{OSTBC} \cdot \mathbf{h}(t) = \begin{pmatrix} \alpha_1 h_{11} + \alpha_2 h_{21} + \alpha_3 h_{31} + \alpha_4 h_{41} \\ \alpha_1 h_{22} - \alpha_2 h_{12} + \alpha_3 h_{42} - \alpha_4 h_{32} \\ \alpha_1 h_{33} - \alpha_2 h_{43} - \alpha_3 h_{13} + \alpha_4 h_{23} \\ \alpha_1 h_{44} + \alpha_2 h_{34} - \alpha_3 h_{24} - \alpha_4 h_{14} \\ \alpha_1^* h_{15} + \alpha_2^* h_{25} + \alpha_3^* h_{35} + \alpha_4^* h_{45} \\ \alpha_1^* h_{26} - \alpha_2^* h_{16} + \alpha_3^* h_{46} - \alpha_4^* h_{36} \\ \alpha_1^* h_{37} - \alpha_2^* h_{47} - \alpha_3^* h_{17} + \alpha_4^* h_{27} \\ \alpha_1^* h_{48} + \alpha_2^* h_{38} - \alpha_3^* h_{28} - \alpha_4^* h_{18} \end{pmatrix}, \tag{3.6}$$

whereas

$$\begin{gathered} h_{11} \neq h_{12} \neq h_{13} \neq h_{14} \neq h_{15} \neq h_{16} \neq h_{17} \neq h_{18} \\ h_{21} \neq h_{22} \neq h_{23} \neq h_{24} \neq h_{25} \neq h_{26} \neq h_{27} \neq h_{28} \\ h_{31} \neq h_{32} \neq h_{33} \neq h_{34} \neq h_{35} \neq h_{36} \neq h_{37} \neq h_{38} \\ h_{41} \neq h_{42} \neq h_{43} \neq h_{44} \neq h_{45} \neq h_{46} \neq h_{47} \neq h_{48} \end{gathered} \tag{3.7}$$

in the general case. Thereby, the first index of h refers to the transmitter and the second index to the time slot. The orthogonality is lost and ISI is introduced, wherefore the decoding has to be adapted as well. A decoding by multiplying with $\mathcal{H}^H$ is no longer possible.

4 Toeplitz STBC

Toeplitz STBC have been first introduced in [4]. Their major benefit is that they achieve full diversity with a linear receiver and at the same time a higher code rate than OSTBCs. Unfortunately, this superior behaviour is only valid for quasi-static flat fading environments.
Motivated to design a linear space-time block code that minimizes the worst case average pairwise error probability (PEP) and which asymptotically achieves the optimal diversity-multiplexing tradeoff derived in [5], the authors use a Toeplitz matrix to convert flat fading MISO channels into a Toeplitz virtual MIMO channel. Thereby, they rely on a Toeplitz matrix $\mathbf{T}(\boldsymbol{\alpha}, N_\mathrm{I}, K)$ generated by a vector ($\boldsymbol{\alpha}$) and a positive integer (K) defined as

$$\mathbf{T}(\boldsymbol{\alpha}, N_\mathrm{I}, K) = \begin{bmatrix} \alpha_1 & 0 & \dots & 0 \\ \alpha_2 & \alpha_1 & \dots & 0 \\ \vdots & \alpha_2 & \ddots & \vdots \\ \alpha_{N_\mathrm{I}} & \ddots & \ddots & \alpha_1 \\ 0 & \ddots & \ddots & \alpha_2 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ddots & 0 & \alpha_{N_\mathrm{I}} \end{bmatrix} \tag{4.1}$$

to generate the code. The Toeplitz space-time block code $\mathbf{X}_\mathrm{Tpltz}(\boldsymbol{\alpha})$, where $\boldsymbol{\alpha}$ is the vector containing the information symbols, is then defined as

$$\mathbf{X}_\mathrm{Tpltz}(\boldsymbol{\alpha}) = \mathbf{T}(\boldsymbol{\alpha}, N, N_\mathrm{TX}) \cdot \mathbf{A}, \tag{4.2}$$

while $\mathbf{A}$ is a $N_\mathrm{TX} \times N_\mathrm{TX}$ invertible matrix which is designed such that the PEP is minimized when a maximum-likelihood (ML) detector is used. However, if the channels are independent, any $N_\mathrm{TX} \times N_\mathrm{TX}$ unitary matrix can be employed. The proposed Toeplitz STBC can achieve a higher code rate R_C than OSTBC, which is

$$R_\mathrm{C,Tpltz} = \frac{N_\mathrm{C} - N_\mathrm{TX} + 1}{N_\mathrm{C}} = 1 - \frac{N_\mathrm{TX} - 1}{N_\mathrm{C}}. \tag{4.3}$$

For a fixed number of transmit antennas N_TX the code-rate R_c can approach 1 if the block-length N_C is sufficiently large, e. g. for $N_\mathrm{TX} = 4$, $R_\mathrm{c}(N_\mathrm{C} = 7) = \frac{4}{7} \approx 57.14\%$

and $R_c(N_C = 32) = \frac{29}{32} \approx 90.63\%$.

The signal on the physical channel $\mathbf{y}_{ch}$ results from the multiplication of the encoding matrix $\mathbf{X}_{Tpltz}(\boldsymbol{\alpha})$ with the channel coefficient matrix $\mathbf{H}$. Assuming independent transmit channels with quasi-static frequency-flat Rayleigh-fading and a MISO setup, the latter becomes a vector $\mathbf{H} \mapsto \mathbf{h}$ with N_{TX} coefficients ($\mathbf{h} = (h_1, h_2, ...h_{N_{TX}})^T$). Neglecting the additive white Gaussian noise (AWGN) on the channel without loss of generality for better readability, the signal on the physical channel can be calculated as

$$\mathbf{y}_{ch} = \mathbf{X}_{Tpltz}(\boldsymbol{\alpha}) \cdot \mathbf{h} = \mathbf{X}_{Tpltz}(\boldsymbol{\alpha}) \cdot \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{N_{TX}} \end{pmatrix}. \tag{4.4}$$

For decoding with a linear receiver, an equivalent channel matrix $\mathcal{H}$ is necessary. For that, first $\widetilde{\mathbf{h}}$ is defined as

$$\widetilde{\mathbf{h}} := \mathbf{A} \cdot \mathbf{h} = \mathbf{A} \cdot \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{N_{TX}} \end{pmatrix} = \begin{pmatrix} \widetilde{h}_1 \\ \widetilde{h}_2 \\ \vdots \\ \widetilde{h}_{N_{TX}} \end{pmatrix}. \tag{4.5}$$

With $\widetilde{\mathbf{h}}$, the equivalent channel model follows as

$$\mathbf{y} = \underbrace{\mathcal{T}(\widetilde{\mathbf{h}}, N_{TX}, N_I)}_{\mathcal{H}} \cdot \boldsymbol{\alpha} = \begin{bmatrix} \widetilde{h}_1 & 0 & \dots & 0 \\ \widetilde{h}_2 & \widetilde{h}_1 & \dots & 0 \\ \vdots & \widetilde{h}_2 & \ddots & \vdots \\ \widetilde{h}_{N_{TX}} & \ddots & \ddots & \widetilde{h}_1 \\ 0 & \ddots & \ddots & \widetilde{h}_2 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ddots & 0 & \widetilde{h}_{N_{TX}} \end{bmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{N_I} \end{pmatrix}. \tag{4.6}$$

E. g. employing a linear *zero-forcing (ZF)* estimator, the received signals $\mathbf{y}$ can be decoded by

$$\hat{\boldsymbol{\alpha}}_{ZF} = (\mathcal{H}^H \mathcal{H})^{-1} \cdot \mathcal{H}^H \cdot \mathbf{y}, \tag{4.7}$$

whereas using a linear *minimum-mean-square-error (MMSE)* estimator, the received signals $\mathbf{y}$ can be decoded by

$$\hat{\boldsymbol{\alpha}}_{MMSE} = \left(\mathbf{I}_{N_I} + \frac{1}{\sigma_n^2} \cdot \mathcal{H}^H \mathcal{H} \right)^{-1} \cdot \mathcal{H}^H \cdot \mathbf{y}. \tag{4.8}$$

It is characteristic for Toeplitz STBC, that the decoding rules stated in Equations 4.7 and 4.8 are sufficient to strive for full diversity. Nonetheless, this property is lost in time-variant scenarios due to similar reasons as for the OSBTCs or the Alamouti code.

4.1 Example

The following example, is based on a MISO setup consisting of four transmit antennas ($N_{\mathrm{TX}} = 4$). Moreover, it is foreseen, that four information symbols are transmitted ($N_{\mathrm{TX}} = 4$), so that $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \alpha_3, \alpha_4]^T$. The block length N_{C} is selected to be $N_{\mathrm{C}} = 7$, so that $N_{\mathrm{I}} = N = N_{\mathrm{C}} - N_{\mathrm{TX}} + 1 = 4$ is fulfilled. Last, independent (uncorrelated) and quasi-static Rayleigh fading transmit channels are assumed, whereas the additive white Gaussian noise (AWGN) is neglected for reasons of simplicity.

For encoding, the *Toeplitz matrix* $\mathcal{T}(\boldsymbol{\alpha}, N_{\mathrm{TX}}, K)$ for the input symbol vector $\boldsymbol{\alpha}$ is necessary as well as a suitable matrix $\mathbf{A}$. Starting with $\mathcal{T}(\boldsymbol{\alpha}, N_{\mathrm{I}}, K)$, the size of the matrix which is defined as $(K+N_{\mathrm{I}}-1)\times K$ can be determined for the given setup with $N_{\mathrm{TX}} = N = N_{\mathrm{C}} - N_{\mathrm{TX}} + 1 = 7-4+1 = 4$ and $K = N_{\mathrm{TX}} = 4$ to be $(4+4-1)\times 4 = 7\times 4$. Then, the particular Toeplitz matrix is

$$\mathcal{T}(\boldsymbol{\alpha}, 4, 4) = \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ \alpha_2 & \alpha_1 & 0 & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 & 0 \\ \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 \\ 0 & \alpha_4 & \alpha_3 & \alpha_2 \\ 0 & 0 & \alpha_4 & \alpha_3 \\ 0 & 0 & 0 & \alpha_4 \end{bmatrix}. \tag{4.9}$$

Since independent (uncorrelated) transmit channels are assumed, any unitary matrix of size $N_{\mathrm{TX}} \times N_{\mathrm{TX}} = 4\times 4$ can be selected for $\mathbf{A}$. However, for better comparability with the example given for the LSDCs in Section 6.4, $\mathbf{A}$ is chosen to be the Fourier-Matrix $\mathbf{F}$, so that

$$\mathbf{A} = \mathbf{F} = \mathrm{fft}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{bmatrix} +1 & +1 & +1 & +1 \\ +1 & -j & -1 & +j \\ +1 & -1 & +1 & -1 \\ +1 & +j & -1 & -j \end{bmatrix}. \tag{4.10}$$

The overall encoding matrix $\mathbf{X}_{\mathrm{Tpltz}}(\boldsymbol{\alpha})$ is the result of the matrix multiplication between $\mathcal{T}(\boldsymbol{\alpha}, N=4, N_{\mathrm{TX}}=4)$ and $\mathbf{A}$. Thus, for the investigated setup $\mathbf{X}_{\mathrm{Tpltz}}(\boldsymbol{\alpha})$ is

$$\mathbf{X}_{\mathrm{Tpltz}} = \mathcal{T}(\boldsymbol{\alpha},4,4)\cdot\mathbf{A} = \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ \alpha_2 & \alpha_1 & 0 & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 & 0 \\ \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 \\ 0 & \alpha_4 & \alpha_3 & \alpha_2 \\ 0 & 0 & \alpha_4 & \alpha_3 \\ 0 & 0 & 0 & \alpha_4 \end{bmatrix} \cdot \begin{bmatrix} +1 & +1 & +1 & +1 \\ +1 & -j & -1 & +j \\ +1 & -1 & +1 & -1 \\ +1 & +j & -1 & -j \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 & \alpha_1 & \alpha_1 & \alpha_1 \\ \alpha_2+\alpha_1 & \alpha_2-j\alpha_1 & \alpha_2-\alpha_1 & \alpha_2+j\alpha_1 \\ \alpha_3+\alpha_2+\alpha_1 & \alpha_3-j\alpha_2-\alpha_1 & \alpha_3-\alpha_2+\alpha_1 & \alpha_3+j\alpha_2-\alpha_1 \\ \alpha_4+\alpha_3+\alpha_2+\alpha_1 & \alpha_4-j\alpha_3-\alpha_2+j\alpha_1 & \alpha_4-\alpha_3+\alpha_2-\alpha_1 & \alpha_4+j\alpha_3-\alpha_2-j\alpha_1 \\ \alpha_4+\alpha_3+\alpha_2 & -j\alpha_4-\alpha_3+j\alpha_2 & -\alpha_4+\alpha_3-\alpha_2 & j\alpha_4-\alpha_3-j\alpha_2 \\ \alpha_4+\alpha_3 & -\alpha_4+j\alpha_3 & \alpha_4-\alpha_3 & -\alpha_4-j\alpha_3 \\ \alpha_4 & j\alpha_4 & -\alpha_4 & -j\alpha_4 \end{bmatrix}. \tag{4.11}$$

So, antenna 1 transmits α_1 in the first time slot, $\alpha_2+\alpha_1$ in the second time slot and so forth until it transmits α_4 in the last time slot. Correspondingly, antenna 2 transmits α_1 in the first time slot, $\alpha_2-j\alpha_1$ in the second time slot and so forth until it transmits $j\alpha_4$ in the last time slot. Antenna 3 and 4 are used to transmit columns 3 and 4 of the matrix.
As it becomes clear from $\mathbf{X}_{\mathrm{Tpltz}}(\boldsymbol{\alpha})$, in total 7 time slots are necessary to transmit 4 information symbols. Hence, the code rate R_{C} is

$$R_{\mathrm{C,Tpltz}} = \frac{4}{7}. \tag{4.12}$$

Assuming independent fading on each transmit channel ($\mathbf{h}=[h_1,h_2,h_3,h_4]^T$) and neglecting the additive white Gaussian noise (AWGN), it follows for the signal on the channel $\mathbf{y}_{\mathrm{ch}}$ that

$$\begin{aligned}\mathbf{y}_{\mathrm{ch}} &= \mathbf{X}_{\mathrm{Tpltz}}\cdot\mathbf{h} \\ &= \begin{bmatrix} \alpha_1h_1+\alpha_1h_2+\alpha_1h_3+\alpha_1h_4 \\ (\alpha_2+\alpha_1)h_1+(\alpha_2-j\alpha_1)h_2+(\alpha_2-\alpha_1)h_3+(\alpha_2+j\alpha_1)h_4 \\ (\alpha_3+\alpha_2+\alpha_1)h_1+(\alpha_3-j\alpha_2-\alpha_1)h_2+(\alpha_3-\alpha_2+\alpha_1)h_3+(\alpha_3+j\alpha_2-\alpha_1)h_4 \\ (\alpha_4+\alpha_3+\alpha_2+\alpha_1)h_1+(\alpha_4-j\alpha_3-\alpha_2+j\alpha_1)h_2+(\alpha_4-\alpha_3+\alpha_2-\alpha_1)h_3\dots \\ \dots+(\alpha_4+j\alpha_3-\alpha_2-j\alpha_1)h_4 \\ (\alpha_4+\alpha_3+\alpha_2)h_1+(-j\alpha_4-\alpha_3+j\alpha_2)h_2+(-\alpha_4+\alpha_3-\alpha_2)h_3+(j\alpha_4-\alpha_3-j\alpha_2)h_4 \\ (\alpha_4+\alpha_3)h_1+(-\alpha_4+j\alpha_3)h_2+(\alpha_4-\alpha_3)h_3+(-\alpha_4-j\alpha_3)h_4 \\ \alpha_4h_1+j\alpha_4h_2+-\alpha_4h_3+-j\alpha_4h_4 \end{bmatrix}.\end{aligned} \tag{4.13}$$

With

$$\widetilde{\mathbf{h}} := \mathbf{A} \cdot \mathbf{h} = \begin{bmatrix} +1 & +1 & +1 & +1 \\ +1 & -j & -1 & +j \\ +1 & -1 & +1 & -1 \\ +1 & +j & -1 & -j \end{bmatrix} \cdot \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix} = \begin{bmatrix} h_1 + h_2 + h_3 + h_4 \\ h_1 - jh_2 - h_3 + jh_4 \\ h_1 - h_2 + h_3 - h_4 \\ h_1 + jh_2 - h_3 - jh_4 \end{bmatrix}, \tag{4.14}$$

$\mathbf{y}_{\text{ch}}$ can be rewritten as

$$\mathbf{y} = \underbrace{\mathcal{T}(\widetilde{\mathbf{h}}, N_{\text{TX}}, N)}_{\mathcal{H}} \cdot \boldsymbol{\alpha} = \begin{bmatrix} \widetilde{\mathbf{h}}(1) & 0 & 0 & 0 \\ \widetilde{\mathbf{h}}(2) & \widetilde{\mathbf{h}}(1) & 0 & 0 \\ \widetilde{\mathbf{h}}(3) & \widetilde{\mathbf{h}}(2) & \widetilde{\mathbf{h}}(1) & 0 \\ \widetilde{\mathbf{h}}(4) & \widetilde{\mathbf{h}}(3) & \widetilde{\mathbf{h}}(2) & \widetilde{\mathbf{h}}(1) \\ 0 & \widetilde{\mathbf{h}}(4) & \widetilde{\mathbf{h}}(3) & \widetilde{\mathbf{h}}(2) \\ 0 & 0 & \widetilde{\mathbf{h}}(4) & \widetilde{\mathbf{h}}(3) \\ 0 & 0 & 0 & \widetilde{\mathbf{h}}(4) \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}, \tag{4.15}$$

where

$$\mathcal{H} = \begin{bmatrix} (h_1 + h_2 + h_3 + h_4) & 0 & 0 & 0 \\ (h_1 - jh_2 - h_3 + jh_4) & (h_1 + h_2 + h_3 + h_4) & 0 & 0 \\ (h_1 - h_2 + h_3 - h_4) & (h_1 - jh_2 - h_3 + jh_4) & (h_1 + h_2 + h_3 + h_4) & 0 \\ (h_1 + jh_2 - h_3 - jh_4) & (h_1 - h_2 + h_3 - h_4) & (h_1 - jh_2 - h_3 + jh_4) & (h_1 + h_2 + h_3 + h_4) \\ 0 & (h_1 + jh_2 - h_3 - jh_4) & (h_1 - h_2 + h_3 - h_4) & (h_1 - jh_2 - h_3 + jh_4) \\ 0 & 0 & (h_1 + jh_2 - h_3 - jh_4) & (h_1 - h_2 + h_3 - h_4) \\ 0 & 0 & 0 & (h_1 + jh_2 - h_3 - jh_4) \end{bmatrix}, \tag{4.16}$$

is the equivalent channel matrix. If maximally time-variant transmit channels are assumed, thus if a different Rayleigh fading channel vector is used for every time-slot, $\mathbf{y}_{\text{ch}}$ can be expressed as

$$\begin{aligned} \mathbf{y}_{\text{ch}} &= \mathbf{X}_{\text{Tpltz}} \cdot \mathbf{h}(t) \\ \mathbf{y}_{\text{ch}}(1) &= \alpha_1(h_{11} + h_{21} + h_{31} + h_{41}) \\ \mathbf{y}_{\text{ch}}(2) &= \alpha_1(h_{12} - jh_{22} - h_{32} + jh_{42}) + \alpha_2(h_{12} + h_{22} + h_{32} + h_{42}) \\ \mathbf{y}_{\text{ch}}(3) &= \alpha_1(h_{13} - h_{23} + h_{33} - h_{43}) + \alpha_2(h_{13} - jh_{23} - h_{33} + jh_{43}) \dots \\ &\quad + \alpha_3(h_{13} + h_{23} + h_{33} + h_{43}) \\ \mathbf{y}_{\text{ch}}(4) &= \alpha_1(h_{14} + jh_{24} - h_{34} - jh_{44}) + \alpha_2(h_{14} - h_{24} + h_{34} - h_{44}) \dots \\ &\quad + \alpha_3(h_{14} - jh_{24} - h_{34} + jh_{44}) + \alpha_4(h_{14} + h_{24} + h_{34} + h_{44}) \\ \mathbf{y}_{\text{ch}}(5) &= \alpha_2(h_{15} + jh_{25} - h_{35} - jh_{45}) + \alpha_3(h_{15} - h_{25} + h_{35} - h_{45}) \dots \\ &\quad + \alpha_4(h_{15} - jh_{25} - h_{35} + jh_{45}) \\ \mathbf{y}_{\text{ch}}(6) &= \alpha_3(h_{16} + jh_{26} - h_{36} - jh_{46}) + \alpha_4(h_{16} - h_{26} + h_{36} - h_{46}) \\ \mathbf{y}_{\text{ch}}(7) &= \alpha_4(h_{17} + jh_{27} - h_{37} - jh_{47}), \end{aligned} \tag{4.17}$$

where i denotes the transmit-antenna and T the time-slot in h_{iT}. Accordingly, $\mathcal{H}$ has to be adopted to

$$\mathcal{H}(:,1) = \begin{pmatrix} (h_{11}+h_{21}+h_{31}+h_{41}) \\ (h_{12}-jh_{22}-h_{32}+jh_{42}) \\ (h_{13}-h_{23}+h_{33}-h_{43}) \\ (h_{14}+jh_{24}-h_{34}-jh_{44}) \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathcal{H}(:,2) = \begin{pmatrix} 0 \\ (h_{12}+h_{22}+h_{32}+h_{42}) \\ (h_{13}-jh_{23}-h_{33}+jh_{43}) \\ (h_{14}-h_{24}+h_{34}-h_{44}) \\ (h_{15}+jh_{25}-h_{35}-jh_{45}) \\ 0 \\ 0 \end{pmatrix}$$
$$\mathcal{H}(:,3) = \begin{pmatrix} 0 \\ 0 \\ (h_{13}+h_{23}+h_{33}+h_{43}) \\ (h_{14}-jh_{24}-h_{34}+jh_{44}) \\ (h_{15}-h_{25}+h_{35}-h_{45}) \\ (h_{16}+jh_{26}-h_{36}-jh_{46}) \\ 0 \end{pmatrix} \quad \mathcal{H}(:,4) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ (h_{14}+h_{24}+h_{34}+h_{44}) \\ (h_{15}-jh_{25}-h_{35}+jh_{45}) \\ (h_{16}-h_{26}+h_{36}-h_{46}) \\ (h_{17}+jh_{27}-h_{37}-jh_{47}) \end{pmatrix}. \tag{4.18}$$

5 Overlapped Alamouti Codes (OAC)

Overlapped Alamouti Codes (OAC) can achieve full-diversity with linear receivers [6] and utilize a similar concept to the previously introduced Toeplitz STBC.
To construct the encoding matrix, first the authors define some ancillary matrices which they denote as $\mathbf{O}(\mathbf{v}, p, q)$ and $\mathbf{E}(\mathbf{v}, p, q)$ of size $(p+q-1) \times q$ where $\mathbf{v}$ is any vector of length p. Furthermore, they use ancillary definitions of $\boldsymbol{\alpha}$ which they refer to as α_o and α_e.
The ancillary matrix $\mathbf{O}(\mathbf{v}, p, q)$ of size $(p+q-1) \times q$ for odd and even q is defined as

$$\mathbf{O}(\mathbf{v}, p, q) = \begin{bmatrix} v_1^* & 0 & \dots & 0 \\ v_2^* & v_1 & \dots & 0 \\ v_3^* & v_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ v_{p-1}^* & v_{p-2} & \dots & v_1^* \\ v_p^* & v_{p-1} & \dots & v_2^* \\ 0 & v_p & \dots & v_3^* \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & v_p^* \end{bmatrix} \text{ and } \begin{bmatrix} v_1^* & 0 & \dots & 0 & 0 \\ v_2^* & v_1 & \dots & 0 & 0 \\ v_3^* & v_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_p^* & v_{p-1} & \dots & v_1^* & 0 \\ 0 & v_p & \dots & v_2^* & v_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & v_p^* & v_{p-1} \\ 0 & 0 & \dots & 0 & v_p \end{bmatrix}, \tag{5.1}$$

while the ancillary matrix $\mathbf{E}(\mathbf{v}, p, q)$ of size $(p+q-1) \times q$ for odd and even q is

$$\mathbf{E}(\mathbf{v}, p, q) = \begin{bmatrix} 0 & \dots & 0 & v_1 \\ 0 & \dots & -v_1^* & v_2 \\ 0 & \dots & -v_2^* & v_3 \\ \vdots & \vdots & \vdots & \vdots \\ v_1 & \dots & -v_{p-2}^* & v_{p-1} \\ v_2 & \dots & v_{p-1}^* & v_p \\ v_3 & \dots & -v_p^* & 0 \\ \vdots & \vdots & \vdots & \vdots \\ v_p & \dots & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & \dots & 0 & v_1 \\ 0 & 0 & \dots & -v_1^* & v_2 \\ 0 & 0 & \dots & -v_2^* & v_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & v_1 & \dots & -v_{p-1}^* & v_p \\ -v_1^* & v_2 & \dots & -v_p^* & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -v_{p-1}^* & v_p & \dots & 0 & 0 \\ -v_p^* & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{5.2}$$

Next, the ancillary vector $\boldsymbol{\alpha}_\mathrm{o}$ which keeps all components of $\boldsymbol{\alpha}$ with odd indexes is

$$\boldsymbol{\alpha}_\mathrm{o} = \begin{cases} (\alpha_1, 0, \alpha_3, 0, \dots, 0, \alpha_{N_\mathrm{I}-1})^T, \; N_\mathrm{TX} \text{ even, } N_\mathrm{I} \text{ even} \\ (\alpha_1, 0, \alpha_3, 0, \dots, 0, \alpha_{N_\mathrm{I}-1}, 0)^T, \; N_\mathrm{TX} \text{ odd, } N_\mathrm{I} \text{ even} \\ (\alpha_1, 0, \alpha_3, 0, \dots, 0, \alpha_{N_\mathrm{I}})^T, \; N_\mathrm{I} \text{ odd} \end{cases} \tag{5.3}$$

whereas the ancillary vector $\boldsymbol{\alpha}_\mathrm{e}$ which keeps all components of $\boldsymbol{\alpha}$ with even indexes is denoted as

$$\boldsymbol{\alpha}_\mathrm{e} = \begin{cases} (\alpha_2, 0, \alpha_4, 0, \dots, 0, \alpha_{N_\mathrm{I}})^T,\ N_\mathrm{TX} \text{ even, } N_\mathrm{I} \text{ even} \\ (\alpha_2, 0, \alpha_4, 0, \dots, 0, \alpha_{N_\mathrm{I}-1}, 0, 0)^T,\ N_\mathrm{TX} \text{ even, } N_\mathrm{I} \text{ odd} \\ (0, \alpha_2, 0, \alpha_4, 0, \dots, 0, \alpha_{N_\mathrm{I}})^T,\ N_\mathrm{TX} \text{ odd, } N_\mathrm{I} \text{ even} \\ (0, \alpha_2, 0, \alpha_4, 0, \dots, 0, \alpha_{N_\mathrm{I}-1}, 0)^T,\ N_\mathrm{TX} \text{ odd, } N_\mathrm{I} \text{ odd} \end{cases}. \tag{5.4}$$

With those definitions in mind, the resulting overlapped Alamouti code matrix can be calculated by

$$\mathbf{X}_\mathrm{OAC}(\boldsymbol{\alpha}) = \begin{cases} \mathbf{O}(\boldsymbol{\alpha}_o, N_\mathrm{I}, N_\mathrm{TX}) + \mathbf{E}(\boldsymbol{\alpha}_e, N_\mathrm{I}, N_\mathrm{TX}),\ N_\mathrm{TX} \text{ odd} \\ \mathbf{O}({\boldsymbol{\alpha}_o}^*, N_\mathrm{I}, N_\mathrm{TX}) + \mathbf{E}(\boldsymbol{\alpha}_e, N_\mathrm{I}, N_\mathrm{TX}),\ N_\mathrm{TX} \text{ even, } N_\mathrm{I} \text{ odd} \\ \mathbf{O}({\boldsymbol{\alpha}_o}^*, N_\mathrm{I}-1, N_\mathrm{TX}) + \mathbf{E}(\boldsymbol{\alpha}_e, N_\mathrm{I}-1, N_\mathrm{TX}),\ N_\mathrm{TX} \text{ even, } N_\mathrm{I} \text{ even} \end{cases}. \tag{5.5}$$

It attracts attention, that each information symbol belongs to two 2×2 Alamouti codes simultaneously except those on the leftmost and rightmost columns. This is why the authors use the term *overlapped* in the naming of their proposed scheme.
Overlapped Alamouti Codes (OAC) can achieve a slightly higher rate R_C than the Toeplitz STBC when N_TX and N_I are even according to

$$R_\mathrm{C} = \begin{cases} \frac{N_\mathrm{I}}{N_\mathrm{I}+N_\mathrm{TX}-2},\ N_\mathrm{TX} \text{ and } N_\mathrm{I} \text{ even} \\ \frac{N_\mathrm{I}}{N_\mathrm{I}+N_\mathrm{TX}-1},\ \text{otherwise} \end{cases} \tag{5.6}$$

It is worth denoting, that the code-rate approaches 1 for a fixed number of transmit antennas N_TX if $N_\mathrm{I} \to \infty$ or respectively $N_\mathrm{C} \to \infty$, e. g. for $N_\mathrm{TX} = 4$, $R_\mathrm{c}(N_\mathrm{C} = 6) = \frac{2}{3} \approx 66.67\%$ and $R_\mathrm{c}(N_\mathrm{C} = 32) = \frac{15}{16} \approx 93.75\%$.
The signal on the physical channel results from the multiplication of the encoding matrix $\mathbf{X}_\mathrm{OAC}(\boldsymbol{\alpha})$ with the channel matrix $\mathbf{H}$. In accordance to the Toeplitz STBC, the latter becomes a vector for independent transmit channels with quasi-static frequency-flat Rayleigh-fading and a MISO setup ($N_\mathrm{RX} = 1$). Thus, $\mathbf{H} \mapsto \mathbf{h}$ with N_TX coefficients, i. e. $\mathbf{h} = (h_1, h_2, ...h_{N_\mathrm{TX}})^T$. Again, neglecting the additive white Gaussian noise (AWGN) on the channel without loss of generality for better readability, the signal on the physical channel $\mathbf{y}_\mathrm{ch}$ can be calculated by

$$\mathbf{y}_\mathrm{ch} = \mathbf{X}_\mathrm{OAC}(\boldsymbol{\alpha}) \cdot \mathbf{h} = \mathbf{X}_\mathrm{OAC}(\boldsymbol{\alpha}) \cdot \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{N_\mathrm{TX}} \end{pmatrix}. \tag{5.7}$$

For decoding with a linear receiver, an equivalent channel matrix $\mathcal{H}$ is necessary, so that the received signal can be described by $\mathbf{y} = \mathcal{H} \cdot \boldsymbol{\alpha}$. For the construction of $\mathcal{H}$ in turn an ancillary matrix, denoted as $\mathbf{F}_{\mathcal{H}}(\mathbf{v}, p, q)$, is employed. Thereby, for odd p and odd and even q $\mathbf{F}_{\mathcal{H}}(\mathbf{v}, p, q)$ is defined as

$$\mathbf{F}_{\mathcal{H}}(\mathbf{v}, p, q) =$$
$$\begin{bmatrix} v_1^* & 0 & \dots & 0 \\ v_2 & v_p & \dots & 0 \\ v_3^* & v_{p-1}^* & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ v_{p-1} & v_3 & \dots & v_1^* \\ v_p^* & -v_2^* & \dots & v_2 \\ 0 & v_1 & \dots & v_3^* \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & v_p^* \end{bmatrix} \text{ and } \begin{bmatrix} v_1^* & 0 & \dots & 0 & 0 \\ v_2 & v_p & \dots & 0 & 0 \\ v_3^* & -v_{p-1}^* & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_p^* & -v_2^* & \dots & v_1^* & 0 \\ 0 & v_1 & \dots & v_2 & v_p \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & v_p^* & -v_2^* \\ 0 & 0 & \dots & 0 & v_1 \end{bmatrix}, \tag{5.8}$$

where for even p and odd and even q $\mathbf{F}_{\mathcal{H}}(\mathbf{v}, p, q)$ is

$$\mathbf{F}_{\mathcal{H}}(\mathbf{v}, p, q) =$$
$$\begin{bmatrix} v_1 & v_p & 0 & 0 & \dots & 0 \\ v_2^* & -v_{p-1}^* & 0 & 0 & \dots & 0 \\ v_3 & v_{p-2} & v_1 & v_p & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_p^* & -v_1^* & v_{p-2}^* & -v_3^* & \dots & v_1 \\ 0 & 0 & v_{p-1} & v_2 & \dots & v_2^* \\ 0 & 0 & v_p^* & -v_1^* & \dots & v_3 \\ \vdots & 0 & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & v_p^* \end{bmatrix} \text{ and } \begin{bmatrix} v_1 & v_p & \dots & 0 & 0 \\ v_2^* & -v_{p-1}^* & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{p-2}^* & -v_3^* & \dots & 0 & 0 \\ v_{p-1} & v_2 & \dots & v_1 & v_p \\ v_p^* & -v_1^* & \dots & v_2^* & -v_{p-1}^* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & v_{p-1} & v_2 \\ 0 & 0 & \dots & v_p^* & -v_1^* \end{bmatrix}. \tag{5.9}$$

With those definitions, $\mathcal{H}$ can be determined in general as

$$\mathcal{H} = (\mathcal{H}_1{}^T, \mathcal{H}_2{}^T, \dots, \mathcal{H}_N{}^T)^T \text{ with } \mathcal{H}_n = \mathbf{F}_{\mathcal{H}}(\mathbf{h}_n, N_{\mathrm{TX}}, N_{\mathrm{I}}). \tag{5.10}$$

Thereby, $\mathbf{h}_n$ is the n-th column of the MIMO channel matrix $\mathbf{H}$. In the MISO case aforementioned definition simplifies to

$$\mathcal{H} = (\mathcal{H}_1{}^T)^T = \mathcal{H}_1 = \mathbf{F}_{\mathcal{H}}(\mathbf{h}, N_{\mathrm{TX}}, N_{\mathrm{I}}). \tag{5.11}$$

It is worth denoting, that all odd columns are orthogonal to all even ones. Thus, the interference between the transmitted symbols is significantly reduced. Moreover, it should be mentioned that the output of the equivalent channel model $\mathbf{y}$ only corresponds to the output of the physical channel $\mathbf{y}_{\mathrm{ch}}$ if the conjugate complex value is used for all even-indexed entries of latter, so that

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \end{pmatrix} = \begin{pmatrix} y_{\mathrm{ch},1} \\ y^*_{\mathrm{ch},2} \\ y_{\mathrm{ch},3} \\ y^*_{\mathrm{ch},4} \\ \vdots \end{pmatrix}. \tag{5.12}$$

In a quasi-static fading environment, OAC can achieve full diversity if a linear receiver is used just like the Toeplitz STBC. However, this property is lost in case of a time-variant fading scenario.

5.1 Example

According to the given example for the Toeplitz STBC in Section 4.1, the following example is also based on a MISO setup consisting of four transmit antennas ($N_{\mathrm{TX}} = 4$). In turn, it is foreseen, that four information symbols are transmitted ($N_{\mathrm{I}} = 4$), so that $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \alpha_3, \alpha_4]^T$.

Starting with the ancillary definitions of $\boldsymbol{\alpha}$, it can be denoted that

$$\begin{aligned} \boldsymbol{\alpha}_o &= (\alpha_1, 0, \alpha_3)^T \\ \boldsymbol{\alpha}_e &= (\alpha_2, 0, \alpha_4)^T \end{aligned} \tag{5.13}$$

With that in mind as well as $q = N_{\mathrm{TX}} = 4$ and $p = N_{\mathrm{I}} - 1 = 3$ the ancillary matrix $\mathbf{E}$ for $\boldsymbol{\alpha}_e$ of size $(p+q-1) \times q = (3+4-1) \times 4 = 6 \times 4$ can be directly constructed as

$$\mathbf{E}(\boldsymbol{\alpha}_e, 3, 4) = \begin{bmatrix} 0 & 0 & 0 & \alpha_{e,1} \\ 0 & 0 & -\alpha^*_{e,1} & \alpha_{e,2} \\ 0 & \alpha_{e,1} & -\alpha^*_{e,2} & \alpha_{e,3} \\ -\alpha^*_{e,1} & \alpha_{e,2} & -\alpha^*_{e,3} & 0 \\ -\alpha^*_{e,2} & \alpha_{e,3} & 0 & 0 \\ -\alpha^*_{e,3} & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \alpha_2 \\ 0 & 0 & -\alpha^*_2 & 0 \\ 0 & \alpha_2 & 0 & \alpha_4 \\ -\alpha^*_2 & 0 & -\alpha^*_4 & 0 \\ 0 & \alpha_4 & 0 & 0 \\ -\alpha^*_4 & 0 & 0 & 0 \end{bmatrix}. \tag{5.14}$$

Similarly, the ancillary matrix $\mathbf{O}$ for $\boldsymbol{\alpha}^*_o = [\alpha^*_1, 0, \alpha^*_3]^T$ of same size can be composed as

$$\mathbf{O}(\boldsymbol{\alpha}^*_o, 3, 4) = \begin{bmatrix} \alpha_{o,1} & 0 & 0 & 0 \\ \alpha_{o,2} & \alpha^*_{o,1} & 0 & 0 \\ \alpha_{o,3} & \alpha^*_{o,2} & \alpha_{o,1} & 0 \\ 0 & \alpha^*_{o,3} & \alpha_{o,2} & \alpha^*_{o,1} \\ 0 & 0 & \alpha_{o,3} & \alpha^*_{o,2} \\ 0 & 0 & 0 & \alpha^*_{o,3} \end{bmatrix} = \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha^*_1 & 0 & 0 \\ \alpha_3 & 0 & \alpha_1 & 0 \\ 0 & \alpha^*_3 & 0 & \alpha^*_1 \\ 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & \alpha^*_3 \end{bmatrix}. \tag{5.15}$$

Lastly, ancillary matrix $\mathbf{F}_{\mathcal{H}}$ which will be necessary to build the equivalent channel matrix for $p = N_{\mathrm{TX}} = 4$, $q = N_{\mathrm{I}} = 4$ and $\mathbf{h} = [h_1, h_2, h_3, h_4]^T$ of size $(q + p - 2) \times 4 = (4 + 4 - 2) \times 4 = 6 \times 4$ follows as

$$\mathbf{F}_{\mathcal{H}}(\mathbf{h}, N_{\mathrm{TX}} = 4, N_{\mathrm{I}} = 4) = \begin{bmatrix} h_1 & h_4 & 0 & 0 \\ h_2^* & -h_3^* & 0 & 0 \\ h_3 & h_2 & h_1 & h_4 \\ h_4^* & -h_1^* & h_2^* & -h_3^* \\ 0 & 0 & h_3 & h_2 \\ 0 & 0 & h_4^* & -h_1^* \end{bmatrix}. \tag{5.16}$$

Using the given construction rules for N_{TX} and N_{I} even, the overall encoding matrix $\mathcal{O}_{N_{\mathrm{TX}},N_{\mathrm{I}}}$ results in

$$\begin{aligned} \mathcal{O}_{N_{\mathrm{TX}},N_{\mathrm{I}}} &= \mathcal{O}_{4,4} = \mathbf{X}_{\mathrm{OAC}}(\boldsymbol{\alpha}) = \mathbf{O}(\boldsymbol{\alpha}_o^*, 3, 4) + \mathbf{E}(\boldsymbol{\alpha}_e, 3, 4) \\ &= \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_1^* & 0 & 0 \\ \alpha_3 & 0 & \alpha_1 & 0 \\ 0 & \alpha_3^* & 0 & \alpha_1^* \\ 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & \alpha_3^* \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \alpha_2 \\ 0 & 0 & -\alpha_2^* & 0 \\ 0 & \alpha_2 & 0 & \alpha_4 \\ -\alpha_2^* & 0 & -\alpha_4^* & 0 \\ 0 & \alpha_4 & 0 & 0 \\ -\alpha_4^* & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 & 0 & 0 & \alpha_2 \\ 0 & \alpha_1^* & -\alpha_2^* & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 & \alpha_4 \\ -\alpha_2^* & \alpha_3^* & -\alpha_4^* & \alpha_1^* \\ 0 & \alpha_4 & \alpha_3 & 0 \\ -\alpha_4^* & 0 & 0 & \alpha_3^* \end{bmatrix}. \end{aligned} \tag{5.17}$$

So, antenna 1 transmits α_1 in the first time-slot, nothing in the second time-slot, α_3 in the third time-slot and so forth until it transmits $-\alpha_4^*$ in the last time-slot. Correspondingly, antenna 2 transmits nothing in the first time-slot, α_1^* in the second time-slot, α_2 in the third time-slot and so forth. Antenna 3 and 4 are used to transmit columns 3 and 4 of the matrix.
As it becomes clear from $\mathcal{O}_{4,4}$, in total 6 time-slots are necessary to transmit 4 information symbols. Hence, the code-rate is

$$R_{\mathrm{C,OAC}} = \frac{4}{6} = \frac{2}{3}. \tag{5.18}$$

Assuming a quasi-static frequency-flat Rayleigh fading transmit channel and neglecting the noise and SNR scaling, the signal on the physical channel can be summarized as

$$\begin{aligned}\mathbf{y}_{\text{ch}} &= \mathbf{X}_{\text{OAC}}(\boldsymbol{\alpha}) \cdot \mathbf{h} \\ &= \begin{bmatrix} \alpha_1 & 0 & 0 & \alpha_2 \\ 0 & \alpha_1^* & -\alpha_2^* & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 & \alpha_4 \\ -\alpha_2^* & \alpha_3^* & -\alpha_4^* & \alpha_1^* \\ 0 & \alpha_4 & \alpha_3 & 0 \\ -\alpha_4^* & 0 & 0 & \alpha_3^* \end{bmatrix} \cdot \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix} = \begin{bmatrix} \alpha_1 h_1 + \alpha_2 h_4 \\ \alpha_1^* h_2 - \alpha_2^* h_3 \\ \alpha_3 h_1 + \alpha_2 h_2 + \alpha_1 h_3 + \alpha_4 h_4 \\ -\alpha_2^* h_1 + \alpha_3^* h_2 - \alpha_4^* h_3 + \alpha_1^* h_4 \\ \alpha_4 h_2 + \alpha_3 h_3 \\ -\alpha_4^* h_1 + \alpha_3^* h_4 \end{bmatrix} \\ &= \begin{bmatrix} \alpha_1 h_1 + \alpha_2 h_4 \\ \alpha_1^* h_2 - \alpha_2^* h_3 \\ \alpha_1 h_3 + \alpha_2 h_2 + \alpha_3 h_1 + \alpha_4 h_4 \\ \alpha_1^* h_4 - \alpha_2^* h_1 + \alpha_3^* h_2 - \alpha_4^* h_3 \\ \alpha_3 h_3 + \alpha_4 h_2 \\ \alpha_3^* h_4 - \alpha_4^* h_1 \end{bmatrix}.\end{aligned} \tag{5.19}$$

Using the equivalent channel matrix $\mathcal{H}$, i. e.

$$\mathcal{H} = \mathbf{F}_{\mathcal{H}}(\mathbf{h}, N_{\text{TX}} = 4, N_{\text{I}} = 4) = \begin{bmatrix} h_1 & h_4 & 0 & 0 \\ h_2^* & -h_3^* & 0 & 0 \\ h_3 & h_2 & h_1 & h_4 \\ h_4^* & -h_1^* & h_2^* & -h_3^* \\ 0 & 0 & h_3 & h_2 \\ 0 & 0 & h_4^* & -h_1^* \end{bmatrix}, \tag{5.20}$$

this relation can also be expressed as

$$\begin{aligned}\mathbf{y} = \mathcal{H} \cdot \boldsymbol{\alpha} &= \begin{bmatrix} h_1 & h_4 & 0 & 0 \\ h_2^* & -h_3^* & 0 & 0 \\ h_3 & h_2 & h_1 & h_4 \\ h_4^* & -h_1^* & h_2^* & -h_3^* \\ 0 & 0 & h_3 & h_2 \\ 0 & 0 & h_4^* & -h_1^* \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ s4 \end{bmatrix} = \begin{bmatrix} \alpha_1 h_1 + \alpha_2 h_4 \\ \alpha_1 h_2^* - \alpha_2 h_3^* \\ \alpha_3 h_1 + \alpha_2 h_2 + \alpha_1 h_3 + \alpha_4 h_4 \\ -\alpha_2 h_1^* + \alpha_3 h_2^* - \alpha_4 h_3^* + \alpha_1 h_4^* \\ \alpha_4 h_2 + \alpha_3 h_3 \\ -\alpha_4 h_1^* + \alpha_3 h_4^* \end{bmatrix} \\ \mathbf{y} &= \begin{bmatrix} \alpha_1 h_1 + \alpha_2 h_4 \\ \alpha_1 h_2^* - \alpha_2 h_3^* \\ \alpha_1 h_3 + \alpha_2 h_2 + \alpha_3 h_1 + \alpha_4 h_4 \\ \alpha_1 h_4^* - \alpha_2 h_1^* + \alpha_3 h_2^* - \alpha_4 h_3^* \\ \alpha_3 h_3 + \alpha_4 h_2 \\ \alpha_3 h_4^* - \alpha_4 h_1^* \end{bmatrix} = \begin{pmatrix} y_{\text{ch},1} \\ y_{\text{ch},2}^* \\ y_{\text{ch},3} \\ y_{\text{ch},4}^* \\ y_{\text{ch},5} \\ y_{\text{ch},6}^* \end{pmatrix}.\end{aligned} \tag{5.21}$$

Thus, the equivalent channel model ($\mathbf{y} = \mathcal{H} \cdot \boldsymbol{\alpha}$) corresponds to the signal on the physical channel if the conjugate complex value is used for all even-indexed entries of the received vector.

Decoding, e. g. with a ZF-receiver can be performed by

$$\hat{\boldsymbol{\alpha}} = (\mathcal{H}^T \cdot \mathcal{H})^{-1} \cdot \mathcal{H}^T \cdot \mathbf{y}. \tag{5.22}$$

If maximally time-varying transmit channels are assumed, i. e. if a different Rayleigh-fading channel coefficient is used for every time-slot, the signal on the physical channel can be expressed as

$$\mathbf{y}_{\text{ch}} = \mathbf{X}_{\text{OAC}} \cdot \mathbf{h}(t) = \begin{bmatrix} \alpha_1 h_{11} + \alpha_2 h_{41} \\ \alpha_1^* h_{22} - \alpha_2^* h_{32} \\ \alpha_1 h_{33} + \alpha_2 h_{23} + \alpha_3 h_{13} + \alpha_4 h_{43} \\ \alpha_1^* h_{44} - \alpha_2^* h_{14} + \alpha_3^* h_{24} - \alpha_4^* h_{34} \\ \alpha_3 h_{35} + \alpha_4 h_{25} \\ \alpha_3^* h_{46} - \alpha_4^* h_{16} \end{bmatrix} \tag{5.23}$$

where i denotes the transmit-antenna and T the time-slot in h_{iT}.

Decoding with a linear ZF-receiver is still possible if the equivalent channel matrix $\mathcal{H} = \mathbf{H}_{\text{eq}}$ is adapted to

$$\mathcal{H} = \begin{bmatrix} h_{11} & h_{41} & 0 & 0 \\ h_{22}^* & -h_{32}^* & 0 & 0 \\ h_{33} & h_{23} & h_{13} & h_{43} \\ h_{44}^* & -h_{14}^* & h_{24}^* & -h_{34}^* \\ 0 & 0 & h_{35} & h_{25} \\ 0 & 0 & h_{46}^* & -h_{16}^* \end{bmatrix}. \tag{5.24}$$

However, the superior diversity performance is lost.

6 Linear Scalable Dispersion Codes

Linear Scalable Dispersion Codes (LSDCs or LSD codes) can achieve both, full diversity and full rate ($R_\mathrm{C} = 1$) at the same time. They were first proposed in [7] in compound with an efficient suboptimal decoder [8]. In principal, LSDCs offer a pure transmit diversity and spatial subchannel mode. Momentarily focusing on single-antenna nodes, the focus is on the former, i. e. the number of subchannels is equal to 1 ($N_\mathrm{U} = 1$). Fundamentally, LSDCs utilize two linear, but decoupled codes which are presented prior to discussing the appropriate communication model.

6.1 Inner Code Construction

The inner code $\mathbf{C}_\nu$ is linear, time-variant and used to adapt to the channel conditions without a-priori channel knowledge. If the inner code was time-invariant, no transmit diversity could be achieved in constant channels because all transmit symbols would be affected by the same equivalent fading variable. In contrast, if the inner code is time-variant, different transmit symbols are affected by different equivalent fading variables. Thus a transmit diversity gain is possible, but typically an appropriate outer code is required.

The simplest foreseen approach is to use antenna switching. Thereby, the different antennas or, respectively, transmitters are activated at different time-slots ν and hence the inner code matrix is built up with orthonormal unit vectors. However, this approach does not allow to utilize an SNR gain in a distributed setup, wherefore it is actually limited to co-located antennas at a single node.

Hence, a more sophisticated inner code matrix can be obtained by utilizing a Fourier matrix $\mathbf{F}$ which the authors claim has empirically proven to be a good choice. For that, the first N_TX columns of the $N_\mathrm{dim} \times N_\mathrm{dim}$ Fourier Matrix $\mathbf{F}$ are used. In other words, the $N_\mathrm{dim} \times N_\mathrm{dim}$ Fourier Matrix $\mathbf{F}$ can be obtained by calculating the fast Fourier transform (FFT) of an $N_\mathrm{dim} \times N_\mathrm{dim}$ identity matrix, so that

$$\mathbf{F} = \mathrm{fft}\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}_{N_\mathrm{dim} \times N_\mathrm{dim}} . \tag{6.1}$$

However, this is still an suboptimal choice. In actuality, the inner code vector has to be carefully chosen, since there can be critical channel realizations.

6.2 Outer Code Construction

The outer code $\mathbf{R}$ is linear, too, but time-invariant and optimized for diversity performance. It is decoupled from the inner code and thus from the channel conditions, so that it does not require any a-priori channel knowledge.
In principal, it is a matrix of size $N_\mathrm{C} \times N_\mathrm{I}$, whereas a $N_\mathrm{I} \times N_\mathrm{I}$ identity matrix can be used for uncoded transmission. To obtain more sophisticated results, the outer code is constructed by optimizing a cost function. For the latter, the authors chose the *averaged pairwise probability of message error (MaxPairMER)*, which is the largest fading averaged error probability of all $(M^{N_\mathrm{I}} - 1) \cdot M^{N_\mathrm{I}}$ pairs of input symbol vectors. It is obvious, that this cost-function depends on the input symbol alphabet whose size is M. Moreover, the authors required the outer code $\mathbf{R}$ to be orthonormal ($\mathbf{R}^H \cdot \mathbf{R} = \mathbf{I}$), so that an orthonormal transformation preserves the Euclidean distance and thus the error performance on an AWGN channel for co-located antennas at a single-node. In addition, the authors preferred the outer code $\mathbf{R}$ to be a cyclic matrix for symmetry and complexity reasons. Considering all these frame conditions, excellent results can be obtained by

$$\mathbf{h}[n] = \frac{1}{\sqrt{N_\mathrm{C}}} \cdot \sum_{k=1}^{N_\mathrm{C}} e^{j \cdot 2\pi \cdot a_\mathrm{cc} \cdot \frac{(k-1)^2}{N_\mathrm{C}^2}} \cdot e^{j \cdot 2\pi \cdot \frac{(k-1)\cdot(n-1)}{N_\mathrm{C}}}. \tag{6.2}$$

where $\mathbf{h} = \mathbf{R}[:,1]$. Besides, $\mathbf{h}$ in this context is the cyclic impulse response of a chirp filter and the parameter a_cc is determined such that the cost function is minimized for given N_I, N_C and input symbol alphabet.
The code rate R_C is determined by N_I and N_C, i. e.

$$R_\mathrm{C,LSDC} = \frac{N_\mathrm{I}}{N_\mathrm{C}}. \tag{6.3}$$

Typically, $N_\mathrm{I} = N_\mathrm{C}$ is chosen, so that full rate can be achieved. However, a simple modification is possible by adding or deleting columns of the outer code matrix. Figuratively speaking, the square matrix which is necessary for a full rate ($R_\mathrm{C} = 1$) code is converted to a rectangular matrix. Apparently, the more rectangular the matrix is, the lower the code rate R_C becomes. Although a full rate is desirable, it can be advantageous to choose $N_\mathrm{C} > N_\mathrm{I}$ to achieve a better BER vs. SNR or, respectively, BER vs. $\frac{E_\mathrm{b}}{N_0}$ performance.

6.3 Communication Model

For encoding, first the input symbol vector $\boldsymbol{\alpha}$ with N_I elements is multiplied with the outer code matrix $\mathbf{R}$ of size $N_\mathrm{C} \times N_\mathrm{I}$ to obtain the transmit symbol vector $\boldsymbol{\alpha}_\mathrm{TX}$ with

N_C elements. This can be expressed as

$$\boldsymbol{\alpha}_\mathrm{TX} = \mathbf{R} \cdot \boldsymbol{\alpha}$$
$$\begin{pmatrix} \alpha_{\mathrm{TX},1} \\ \alpha_{\mathrm{TX},2} \\ \vdots \\ \alpha_{\mathrm{TX},N_\mathrm{C}} \end{pmatrix} = \begin{bmatrix} R_{1,1} & R_{1,2} & \dots & R_{1,N_\mathrm{I}} \\ R_{2,1} & R_{2,2} & \dots & R_{2,N_\mathrm{I}} \\ \vdots & \vdots & \vdots & \vdots \\ R_{N_\mathrm{C},1} & R_{N_\mathrm{C},2} & \dots & R_{N_\mathrm{C},N_\mathrm{I}} \end{bmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{N_\mathrm{I}} \end{pmatrix}. \tag{6.4}$$

In each time-slot ν one element of $\boldsymbol{\alpha}_\mathrm{TX}$ is used for further processing. Thereby, each element is multiplied with the corresponding inner code vector $\mathbf{c}_\nu$. Thus, the transmit symbols can be given as

$$\begin{aligned} \boldsymbol{\alpha}_{\nu=1} &= \mathbf{c}_1 \cdot \alpha_{\mathrm{TX},1} \\ \boldsymbol{\alpha}_{\nu=2} &= \mathbf{c}_2 \cdot \alpha_{\mathrm{TX},2} \\ \vdots &= \vdots \\ \boldsymbol{\alpha}_{\nu=N_\mathrm{C}} &= \mathbf{c}_{N_\mathrm{C}} \cdot \alpha_{\mathrm{TX},N_\mathrm{C}}. \end{aligned} \tag{6.5}$$

With that the overall LSDC encoding matrix can be composed as

$$\mathbf{X}_\mathrm{LSDC} = \begin{bmatrix} (\boldsymbol{\alpha}_{\nu=1})^T \\ (\boldsymbol{\alpha}_{\nu=2})^T \\ \vdots \\ (\boldsymbol{\alpha}_{\nu=N_\mathrm{C}})^T \end{bmatrix}. \tag{6.6}$$

Assuming independent transmit channels with quasi-static frequency-flat Rayleigh-fading and neglecting the noise on the channel without loss of generality for better readability, the signal on the physical channel can be calculated by

$$\mathbf{y}_\mathrm{ch} = \mathbf{X}_\mathrm{LSDC} \cdot \mathbf{h} = \mathbf{X}_\mathrm{LSDC} \cdot \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{N_\mathrm{TX}} \end{pmatrix}. \tag{6.7}$$

In order to decode the received symbols, it is necessary to express this relation with an equivalent channel model, so that a matched received symbol vector $\mathbf{y}_\mathrm{m}$ can also be described as

$$\mathbf{y}_\mathrm{m} = \boldsymbol{\Lambda}_\mathrm{ISI} \cdot \boldsymbol{\alpha}. \tag{6.8}$$

Thereby, $\boldsymbol{\Lambda}_\mathrm{ISI}$ denotes a correlation matrix which summarizes the encoding with the outer and inner code, the impact of the channel and the corresponding equalization. As an intermediate step, $\boldsymbol{\Lambda}_\mathrm{ISI}$ can be expressed as

$$\boldsymbol{\Lambda}_\mathrm{ISI} = \mathbf{R}^H \cdot \mathbf{D} \cdot \mathbf{R}. \tag{6.9}$$

$\mathbf{D}$ in turn is a diagonal matrix whose elements can be calculated by

$$\mathbf{D}(\nu,\nu) = \mathbf{c}_\nu^H \cdot \mathbf{h} \cdot \mathbf{h}^H \cdot \mathbf{c}_\nu = ||\mathbf{h}^H \cdot \mathbf{c}_\nu||^2$$
$$\mathbf{D} = \begin{bmatrix} ||\mathbf{h}^H \cdot \mathbf{c}_1||^2 & 0 & \dots & 0 \\ 0 & ||\mathbf{h}^H \cdot \mathbf{c}_2||^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & ||\mathbf{h}^H \cdot \mathbf{c}_{N_\mathrm{TX}}||^2 \end{bmatrix}. \tag{6.10}$$

The diagonal elements of $\mathbf{D}$ are the equivalent fading variables utilized by the outer code. Moreover, it becomes obvious from this description that for each time-slot (represented as a row of the matrix) a different channel vector $\mathbf{h} \mapsto \mathbf{h}_\nu$ can be used if time-variant channels are assumed. Thus, LSD codes directly consider this kind of fading scenario in their design, wherefore no performance degradation is to be expected.

The fading on the transmit channels causes intersymbol interference (ISI), whereas the latter is linear and represented by $\mathbf{\Lambda}_\mathrm{ISI}$. To compensate this ISI a decoder using an equalization method is needed. Naturally, a maximum-likelihood (ML) decoder is optimal for that interference cancellation. However, the complexity of the ML-decoder increases exponentially with N_I and the input symbol alphabet size M, wherefore such a receiver architecture is only feasible for small N_I and M. Consequently, only suboptimal methods are of practical use and due to the linearity of the code many different known ISI decoders can be applied. These all have in common, that they are suboptimal and introduce a tradeoff between complexity and performance. Additionally to the existing decoders, the authors proposed a further suboptimal reduced complexity decoder which they refer to as *maximum a-posteriori minimum mean-square error decision feedback equalizer (MAP-MMSE-DFE)*. However, the received symbol vector $\mathbf{y}_\mathrm{ch} = \mathbf{y}_\mathrm{rx}$ cannot be directly forwarded to the decoder, but has to be correlated before. Accordingly, each element of $\mathbf{y}_\mathrm{rx}$ is first multiplied with the matched channel vector $\mathbf{h}^H$ and thereafter with the matched corresponding inner code vector $\mathbf{c}_\nu$. Finally, the channel and inner code correlated received symbol vector is multiplied with the matched outer code matrix $\mathbf{R}^H$ to obtain the overall matched received symbol vector $\mathbf{y}_\mathrm{m}$ which then can be decoded with the MAP-MMSE-DFE. This proceeding can be denoted as

$$\mathbf{y}_\mathrm{m} = \mathbf{R}^H \cdot \begin{pmatrix} \mathbf{c}_1^H \cdot \mathbf{h}^H \cdot y_{\mathrm{ch},1} \\ \mathbf{c}_2^H \cdot \mathbf{h}^H \cdot y_{\mathrm{ch},2} \\ \vdots \\ \mathbf{c}_{N_\mathrm{TX}}^H \cdot \mathbf{h}^H \cdot y_{\mathrm{ch},N_\mathrm{C}} \end{pmatrix}. \tag{6.11}$$

In addition to the correlated received symbol vector $\mathbf{y}_\mathrm{m}$, the employed MAP-MMSE-DFE requires knowledge about the correlation matrix $\mathbf{\Lambda}_\mathrm{ISI}$, the variance of the input symbols σ_α^2, the noise variance σ_n^2, the number of the input symbol vector elements N_I and the user defined decision threshold $\mathrm{p}_{\mathrm{e,THRES}}$. Thereby, the correlation matrix $\mathbf{\Lambda}_{\mathrm{ISI}}$ has to be calculated at the receiver using the obtained knowledge about the channel and the a-priori available knowledge about the outer and inner code. The variance of the input symbols σ_α^2 is typically also known a-priori at the receiver just like the number of the input symbol vector elements N_I. Normally, the noise variance σ_n^2 cannot be known a-priori which is why it has to be estimated. Lastly, the user defined decision error probability threshold $\mathrm{p}_{\mathrm{e,THRES}}$ has to be set. If $\mathrm{p}_{\mathrm{e,THRES}} = 0$, in each iteration, only one symbol is decoded. In contrast, if $\mathrm{p}_{\mathrm{e,THRES}}$ is increased, the complexity can be reduced equivalent to that of a linear MMSE-receiver.

6.4 Example

To match with the given examples for Toeplitz STBC and Overlapped Alamouti Codes in Sections 4.1 and 5.1 respectively, the following example is also based on a MISO setup consisting of four transmit antennas ($N_{\mathrm{TX}} = 4$ and $N_{\mathrm{RX}} = 1$). For reasons of better comparability, it is foreseen to transmit four information symbols in four time slots, which will enable full code rate ($N_\mathrm{c} = N_\mathrm{I} = 4$). Lastly, independent, quasi-static frequency-flat Rayleigh fading transmit channels are assumed.

Because $N_\mathrm{c} = N_\mathrm{I} = 4$, the outer code matrix $\mathbf{R}$ becomes a 4×4 square matrix. One numerically optimized complex solution is

$$\begin{aligned}\mathbf{R} &= \begin{bmatrix} R_{1,1} & R_{1,2} & R_{1,3} & R_{1,4} \\ R_{2,1} & R_{2,2} & R_{2,3} & R_{2,4} \\ R_{3,1} & R_{3,2} & R_{3,3} & R_{3,4} \\ R_{4,1} & R_{4,2} & R_{4,3} & R_{4,4} \end{bmatrix} \\ &= \begin{bmatrix} 0.3192 - 0.0001j & 0.0496 + 0.6597j & 0.4078 - 0.4454j & 0.2234 - 0.2142j \\ 0.0496 + 0.6597j & 0.4078 - 0.4454j & 0.2234 - 0.2142j & 0.3192 - 0.0001j \\ 0.4078 - 0.4454j & 0.2234 - 0.2142j & 0.3192 - 0.0001j & 0.0496 + 0.6597j \\ 0.2234 - 0.2142j & 0.3192 - 0.0001j & 0.0496 + 0.6597j & 0.4078 - 0.4454j \end{bmatrix}.\end{aligned} \tag{6.12}$$

For reasons of simplicity, in the following the indexed writing of the individual elements will be used rather than the concrete numeric values.
As already mentioned in the code construction, one empirically good choice for the inner code matrix $\mathbf{C}_\nu$ is the Fourier matrix, i. e.

$$\mathbf{C}_\nu = \mathbf{F} = \mathrm{fft}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{bmatrix} +1 & +1 & +1 & +1 \\ +1 & -j & -1 & +j \\ +1 & -1 & +1 & -1 \\ +1 & +j & -1 & -j \end{bmatrix}. \tag{6.13}$$

In each time slot ν one column of the Fourier matrix $\mathbf{C}_\nu$ is used and represents the inner code vector $\mathbf{c}_\nu$. Hence, there is a different inner code vector for each time slot, the inner code is time-variant. In total 4 time slots are necessary to transmit the 4 information symbols, so that the inner code vector $\mathbf{c}_\nu$ for each time slot ($\nu = 1, 2, 3, 4$) can be denoted as

$$\mathbf{c}_1 = \begin{pmatrix} +1 \\ +1 \\ +1 \\ +1 \end{pmatrix};\ \mathbf{c}_2 = \begin{pmatrix} +1 \\ -j \\ -1 \\ +j \end{pmatrix};\ \mathbf{c}_3 = \begin{pmatrix} +1 \\ -1 \\ +1 \\ -1 \end{pmatrix};\ \mathbf{c}_4 = \begin{pmatrix} +1 \\ +j \\ -1 \\ -j \end{pmatrix}. \tag{6.14}$$

First of all, the input symbol vector $\boldsymbol{\alpha}$ which contains the information symbols is multiplied with the outer-code matrix. The result, referred to as $\boldsymbol{\alpha}_{\mathrm{TX}}$, is then

$$\begin{aligned} \boldsymbol{\alpha}_{\mathrm{TX}} &= \mathbf{R} \cdot \boldsymbol{\alpha} \\ &= \begin{bmatrix} R_{1,1} & R_{1,2} & R_{1,3} & R_{1,4} \\ R_{2,1} & R_{2,2} & R_{2,3} & R_{2,4} \\ R_{3,1} & R_{3,2} & R_{3,3} & R_{3,4} \\ R_{4,1} & R_{4,2} & R_{4,3} & R_{4,4} \end{bmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} R_{1,1} \cdot \alpha_1 + R_{1,2} \cdot \alpha_2 + R_{1,3} \cdot \alpha_3 + R_{1,4} \cdot \alpha_4 \\ R_{2,1} \cdot \alpha_1 + R_{2,2} \cdot \alpha_2 + R_{2,3} \cdot \alpha_3 + R_{2,4} \cdot \alpha_4 \\ R_{3,1} \cdot \alpha_1 + R_{3,2} \cdot \alpha_2 + R_{3,3} \cdot \alpha_3 + R_{3,4} \cdot \alpha_4 \\ R_{4,1} \cdot \alpha_1 + R_{4,2} \cdot \alpha_2 + R_{4,3} \cdot \alpha_3 + R_{4,4} \cdot \alpha_4 \end{pmatrix}. \end{aligned} \tag{6.15}$$

In each time-slot $\nu = 1, 2, 3, 4$ one symbol of $\boldsymbol{\alpha}_{\mathrm{TX}}$ is transmitted. Hence, it can be denoted that

$$\begin{aligned} \alpha_{\mathrm{TX},1} &= R_{1,1} \cdot \alpha_1 + R_{1,2} \cdot \alpha_2 + R_{1,3} \cdot \alpha_3 + R_{1,4} \cdot \alpha_4, \\ \alpha_{\mathrm{TX},2} &= R_{2,1} \cdot \alpha_1 + R_{2,2} \cdot \alpha_2 + R_{2,3} \cdot \alpha_3 + R_{2,4} \cdot \alpha_4, \\ \alpha_{\mathrm{TX},3} &= R_{3,1} \cdot \alpha_1 + R_{3,2} \cdot \alpha_2 + R_{3,3} \cdot \alpha_3 + R_{3,4} \cdot \alpha_4 \text{ and} \\ \alpha_{\mathrm{TX},4} &= R_{4,1} \cdot \alpha_1 + R_{4,2} \cdot \alpha_2 + R_{4,3} \cdot \alpha_3 + R_{4,4} \cdot \alpha_4. \end{aligned} \tag{6.16}$$

Thereafter, the symbols to be transmitted $\boldsymbol{\alpha}_\nu$ in the ν-th time-slot can be obtained by multiplying $\boldsymbol{\alpha}_{\mathrm{TX}}$ with the corresponding inner-code vector $\mathbf{c}_\nu$, i. e.

$$\begin{aligned} \boldsymbol{\alpha}_{\nu=1} &= \alpha_{\mathrm{TX},1} \cdot \mathbf{c}_1 = \alpha_{\mathrm{TX},1} \cdot \begin{pmatrix} +1 \\ +1 \\ +1 \\ +1 \end{pmatrix} = \begin{pmatrix} \alpha_{\nu=1,1} \\ \alpha_{\nu=1,2} \\ \alpha_{\nu=1,3} \\ \alpha_{\nu=1,4} \end{pmatrix}, \\ \boldsymbol{\alpha}_{\nu=2} &= \alpha_{\mathrm{TX},2} \cdot \mathbf{c}_2 = \alpha_{\mathrm{TX},2} \cdot \begin{pmatrix} +1 \\ -j \\ -1 \\ +j \end{pmatrix} = \begin{pmatrix} \alpha_{\nu=2,1} \\ \alpha_{\nu=2,2} \\ \alpha_{\nu=2,3} \\ \alpha_{\nu=2,4} \end{pmatrix}, \\ \boldsymbol{\alpha}_{\nu=3} &= \alpha_{\mathrm{TX},3} \cdot \mathbf{c}_3 = \alpha_{\mathrm{TX},3} \cdot \begin{pmatrix} +1 \\ -1 \\ +1 \\ -1 \end{pmatrix} = \begin{pmatrix} \alpha_{\nu=3,1} \\ \alpha_{\nu=3,2} \\ \alpha_{\nu=3,3} \\ \alpha_{\nu=3,4} \end{pmatrix} \text{ and} \end{aligned} \tag{6.17}$$

$$\boldsymbol{\alpha}_{\nu=4} = \alpha_{\mathrm{TX},4} \cdot \mathbf{c}_4 = \alpha_{\mathrm{TX},4} \cdot \begin{pmatrix} +1 \\ +j \\ -1 \\ -j \end{pmatrix} = \begin{pmatrix} \alpha_{\nu=4,1} \\ \alpha_{\nu=4,2} \\ \alpha_{\nu=4,3} \\ \alpha_{\nu=4,4} \end{pmatrix}. \tag{6.18}$$

Finally, the overall encoding matrix $\mathbf{X}_{\mathrm{LSDC}}$ can be summarized as

$$\mathbf{X}_{\mathrm{LSDC}}(\boldsymbol{\alpha}) = \begin{bmatrix} (\boldsymbol{\alpha}_{\nu=1})^T \\ (\boldsymbol{\alpha}_{\nu=2})^T \\ (\boldsymbol{\alpha}_{\nu=3})^T \\ (\boldsymbol{\alpha}_{\nu=4})^T \end{bmatrix} = \begin{bmatrix} \alpha_{\nu=1,1} & \alpha_{\nu=1,2} & \alpha_{\nu=1,3} & \alpha_{\nu=1,4} \\ \alpha_{\nu=2,1} & \alpha_{\nu=2,2} & \alpha_{\nu=2,3} & \alpha_{\nu=2,4} \\ \alpha_{\nu=3,1} & \alpha_{\nu=3,2} & \alpha_{\nu=3,3} & \alpha_{\nu=3,4} \\ \alpha_{\nu=4,1} & \alpha_{\nu=4,2} & \alpha_{\nu=4,3} & \alpha_{\nu=4,4} \end{bmatrix}. \tag{6.19}$$

The signal on the physical channel results from the multiplication of the encoding matrix $\mathbf{X}_{\mathrm{LSDC}}$ with the channel vector $\mathbf{h} = (h_1, h_2, ...h_{N_{\mathrm{TX}}})^T$. Neglecting the additive white Gaussian noise (AWGN) on the channel without loss of generality for better readability, the signal on the physical channel can be calculated as

$$\mathbf{y}_{\mathrm{ch}} = \mathbf{X}_{\mathrm{LSDC}} \cdot \mathbf{h} = \mathbf{X}_{\mathrm{LSDC}} \cdot \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix} = \begin{pmatrix} y_{\mathrm{ch},1} \\ y_{\mathrm{ch},2} \\ y_{\mathrm{ch},3} \\ y_{\mathrm{ch},4} \end{pmatrix}. \tag{6.20}$$

Regardless of the used decoder, the received symbol vector $\mathbf{y}_{\mathrm{ch}} = \mathbf{y}_{\mathrm{rx}}$ cannot be directly forwarded, but has to be correlated before. For this example, this correlation process can be denoted as

$$\mathbf{y}_{\mathrm{m}} = \mathbf{R}^H \cdot \begin{pmatrix} \mathbf{c}_1^H \cdot \mathbf{h}^H \cdot y_{\mathrm{ch},1} \\ \mathbf{c}_2^H \cdot \mathbf{h}^H \cdot y_{\mathrm{ch},2} \\ \mathbf{c}_3^H \cdot \mathbf{h}^H \cdot y_{\mathrm{ch},3} \\ \mathbf{c}_4^H \cdot \mathbf{h}^H \cdot y_{\mathrm{ch},4} \end{pmatrix} = \begin{bmatrix} R_{1,1}^* & R_{2,1}^* & R_{3,1}^* & R_{4,1}^* \\ R_{1,2}^* & R_{2,2}^* & R_{3,2}^* & R_{4,2}^* \\ R_{1,3}^* & R_{2,3}^* & R_{3,3}^* & R_{4,3}^* \\ R_{1,4}^* & R_{2,4}^* & R_{3,4}^* & R_{4,4}^* \end{bmatrix} \cdot \begin{pmatrix} \mathbf{c}_1^H \cdot \mathbf{h}^H \cdot y_{\mathrm{ch},1} \\ \mathbf{c}_2^H \cdot \mathbf{h}^H \cdot y_{\mathrm{ch},2} \\ \mathbf{c}_3^H \cdot \mathbf{h}^H \cdot y_{\mathrm{ch},3} \\ \mathbf{c}_4^H \cdot \mathbf{h}^H \cdot y_{\mathrm{ch},4} \end{pmatrix}. \tag{6.21}$$

Accordingly, the correlation matrix $\mathbf{\Lambda}_{\mathrm{ISI}}$ follows as

$$\begin{aligned} \mathbf{D}(\nu,\nu) &= \mathbf{c}_\nu^H \cdot \mathbf{h} \cdot \mathbf{h}^H \cdot \mathbf{c}_\nu = ||\mathbf{h}^H \cdot \mathbf{c}_\nu||^2 \\ \mathbf{D} &= \begin{bmatrix} ||\mathbf{h}^H \cdot \mathbf{c}_1||^2 & 0 & 0 & 0 \\ 0 & ||\mathbf{h}^H \cdot \mathbf{c}_2||^2 & 0 & 0 \\ 0 & 0 & ||\mathbf{h}^H \cdot \mathbf{c}_3||^2 & 0 \\ 0 & 0 & 0 & ||\mathbf{h}^H \cdot \mathbf{c}_4||^2 \end{bmatrix}. \end{aligned} \tag{6.22}$$

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