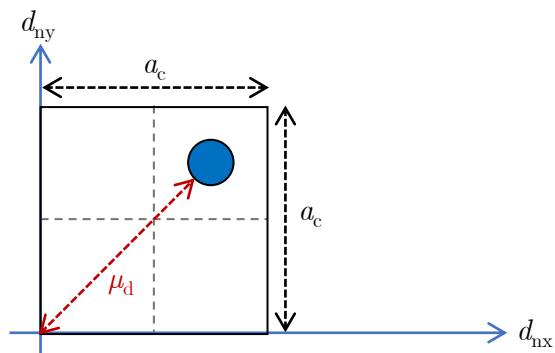


- White Paper -  
Expected Mean Distance in a 2D Square Cluster  
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# 1 Problem Statement

Within this paper, we are basically interested in the expected mean distance of a randomly placed node within a square cluster to the origin that we denote as  $\mu_d$ . Thereby, we concentrate on a 2-dimensional (2D) square cluster of size  $a_c \times a_c$  as depicted on the titlepage. Furthermore, we presume a uniform distribution for the x- and y-coordinate in a Cartesian coordinate system. That is, both, the x- as well as y-coordinate are independent and uniformly randomly distributed between  $[0, a_c]$ . Consequently,  $\mu_d$  can be generally expressed by

$$\mu_d = \int_0^{a_c} \int_0^{a_c} \sqrt{d_{nx}^2 + d_{ny}^2} \cdot f_{d_{nx}, d_{ny}}(d_{nx}, d_{ny}) dx dy, \quad (1.1)$$

where

$$f_{d_{nx}, d_{ny}}(d_{nx}, d_{ny}) = f_{d_{nx}}(d_{nx}) \cdot f_{d_{ny}}(d_{ny}) = \frac{1}{a_c} \cdot \frac{1}{a_c} = \frac{1}{a_c^2}, \quad (1.2)$$

denotes the probability density function for the nodes' position. Accordingly, the initial integral can be formulated as

$$\mu_d = \frac{1}{a_c^2} \cdot \int_0^{a_c} \int_0^{a_c} \sqrt{d_{nx}^2 + d_{ny}^2} dx dy. \quad (1.3)$$

Subsequently, we analytically solve this integral, whereas we use  $d_{nx} = x$  and  $d_{ny} = y$  for the sake of clarity in our writing for the remaining. The overall simplified solution, i. e.  $\mu_d$  with respect to the cluster dimension  $a_c$ , can be stated as

$$\mu_d = \frac{1}{a_c^2} \cdot \int_0^{a_c} \int_0^{a_c} \sqrt{x^2 + y^2} dx dy = a_c \cdot \frac{1}{3} \cdot [\ln(1 + \sqrt{2}) + \sqrt{2}] \quad (1.4)$$

which allows to determine the average distance of a randomly placed node within a square cluster of given size  $a_c$ .

## 2 Inner Integral

Starting with the inner integral, i. e.

$$\int_0^{a_c} \sqrt{x^2 + y^2} dx, \quad (2.1)$$

we first employ a trigonometric substitution, where

$$\begin{aligned} x = y \cdot \tan(\phi) &\rightarrow \phi = \arctan\left(\frac{x}{y}\right) \\ \frac{dx}{d\phi} = y \cdot \frac{1}{\cos^2(\phi)} &\rightarrow dx = y \cdot \frac{1}{\cos^2(\phi)} d\phi, \end{aligned} \quad (2.2)$$

so, that the integral can be written as

$$\begin{aligned} &\int_0^{\arctan\left(\frac{a_c}{y}\right)} \sqrt{(y \cdot \tan(\phi))^2 + y^2} \cdot y \cdot \frac{1}{\cos^2(\phi)} d\phi \\ &= \int_0^{\arctan\left(\frac{a_c}{y}\right)} \sqrt{y^2 \cdot (1 + \tan^2(\phi))} \cdot y \cdot \frac{1}{\cos^2(\phi)} d\phi = y^2 \cdot \int_0^{\arctan\left(\frac{a_c}{y}\right)} \frac{1}{\cos^3(\phi)} d\phi, \end{aligned} \quad (2.3)$$

whereas we used the identity  $1 + \tan^2(\phi) = \frac{1}{\cos^2(\phi)}$ . Focusing on the integral of  $\cos^{-3}(\phi)$  we can denote

$$\int_0^{\arctan\left(\frac{a_c}{y}\right)} \frac{1}{\cos^3(\phi)} d\phi = \int_0^{\arctan\left(\frac{a_c}{y}\right)} \frac{1}{\cos(\phi)} \cdot \frac{1}{\cos^2(\phi)} d\phi. \quad (2.4)$$

Subsequently, we will employ integration by parts, whereas

$$\begin{aligned} u(\phi) = \tan(\phi) &\rightarrow \frac{du}{d\phi} = \frac{1}{\cos^2(\phi)} \quad \text{and} \\ v(\phi) = \frac{1}{\cos(\phi)} &\rightarrow \frac{dv}{d\phi} = \frac{\sin(\phi)}{\cos^2(\phi)}. \end{aligned} \quad (2.5)$$

Accordingly, it can be noted that

$$\begin{aligned}
 \int \frac{1}{\cos^3(\phi)} d\phi &= \int \frac{1}{\cos(\phi)} \cdot \frac{1}{\cos^2(\phi)} d\phi \\
 &= \tan(\phi) \cdot \frac{1}{\cos(\phi)} - \int \tan(\phi) \cdot \frac{\sin(\phi)}{\cos^2(\phi)} d\phi = \frac{\tan(\phi)}{\cos(\phi)} - \int \frac{\sin(\phi)}{\cos(\phi)} \cdot \frac{\sin(\phi)}{\cos^2(\phi)} d\phi \\
 &= \frac{\tan(\phi)}{\cos(\phi)} - \int \frac{\sin^2(\phi)}{\cos^2(\phi)} \cdot \frac{1(\phi)}{\cos(\phi)} d\phi = \frac{\tan(\phi)}{\cos(\phi)} - \int \tan^2(\phi) \cdot \frac{1(\phi)}{\cos(\phi)} d\phi \\
 &= \frac{\tan(\phi)}{\cos(\phi)} - \int \left( \frac{1}{\cos^2(\phi)} - 1 \right) \cdot \frac{1(\phi)}{\cos(\phi)} d\phi = \frac{\tan(\phi)}{\cos(\phi)} - \int \frac{1}{\cos^3(\phi)} d\phi + \int \frac{1}{\cos(\phi)} d\phi \\
 &= \frac{1}{2} \cdot \frac{\tan(\phi)}{\cos(\phi)} + \frac{1}{2} \cdot \int \frac{1}{\cos(\phi)} d\phi.
 \end{aligned} \tag{2.6}$$

We did not denote the integral borders for the sake of clarity, which we catch up, now. Thus, we have to write

$$\begin{aligned}
 \int_0^{\arctan(\frac{a_c}{y})} \frac{1}{\cos^3(\phi)} d\phi &= \frac{1}{2} \cdot \left( \int_0^{\arctan(\frac{a_c}{y})} \frac{1}{\cos(\phi)} d\phi + \frac{\tan(\arctan(\frac{a_c}{y}))}{\cos(\arctan(\frac{a_c}{y}))} \right) \\
 &= \frac{1}{2} \cdot \left( \int_0^{\arctan(\frac{a_c}{y})} \frac{1}{\cos(\phi)} d\phi + \frac{a_c}{y} \cdot \frac{1}{\cos(\arctan(\frac{a_c}{y}))} \right).
 \end{aligned} \tag{2.7}$$

With  $\cos(\arctan(x)) = \frac{1}{\sqrt{x^2+1}}$  this can be also written as

$$\begin{aligned}
 \int_0^{\arctan(\frac{a_c}{y})} \frac{1}{\cos^3(\phi)} d\phi &= \frac{1}{2} \cdot \left( \int_0^{\arctan(\frac{a_c}{y})} \frac{1}{\cos(\phi)} d\phi + \frac{a_c}{y} \cdot \sqrt{\left(\frac{a_c}{y}\right)^2 + 1} \right) \\
 &= \frac{1}{2} \cdot \left( \int_0^{\arctan(\frac{a_c}{y})} \frac{1}{\cos(\phi)} d\phi + \left(\frac{a_c}{y}\right)^2 \cdot \sqrt{\left(\frac{y}{a_c}\right)^2 + 1} \right).
 \end{aligned} \tag{2.8}$$

Next, we have to solve the integral of  $\cos^{-1}(\phi)$ , whereas we can formulate the problem as

$$\int \frac{1}{\cos(\phi)} d\phi = \int \frac{\cos(\phi)}{\cos^2(\phi)} d\phi. \tag{2.9}$$

We in turn employ substitution, so that

$$\begin{aligned}
 \phi &= \sin(\theta) \rightarrow \theta = \arcsin(\phi) \\
 \frac{d\theta}{d\phi} &= \cos(\phi) \rightarrow d\phi = \frac{d\theta}{\cos(\phi)}.
 \end{aligned} \tag{2.10}$$

Besides, we use the identity  $\cos^2(\phi) = 1 - \sin^2(\phi)$ . With that, we can write

$$\int \frac{1}{\cos(\phi)} d\phi = \int \frac{\cos(\phi)}{\cos^2(\phi)} d\phi = \int \frac{\cos(\phi)}{1 - \theta^2} \cdot \frac{d\theta}{\cos(\phi)} = \int \frac{1}{1 - \theta^2} d\theta. \quad (2.11)$$

We need partial fraction expansion to solve the integral. Hence,

$$\begin{aligned} \int \frac{1}{1 - \theta^2} d\theta &= \int -1 \cdot \left( \frac{1}{2} \cdot \frac{1}{\theta - 1} - \frac{1}{2} \cdot \frac{1}{\theta + 1} \right) d\theta = \int \frac{1}{2} \cdot \frac{1}{\theta + 1} - \frac{1}{2} \cdot \frac{1}{\theta - 1} d\theta \\ &= \frac{1}{2} \cdot \int \frac{1}{\theta + 1} - \frac{1}{\theta - 1} d\theta = \frac{1}{2} \cdot (\ln(|\theta + 1|) - \ln(|\theta - 1|)) = \frac{1}{2} \cdot \ln \left( \left| \frac{\theta + 1}{\theta - 1} \right| \right). \end{aligned} \quad (2.12)$$

Resubstituting delivers

$$\int \frac{1}{\cos(\phi)} d\phi = \frac{1}{2} \cdot \ln \left( \left| \frac{\sin(\phi) + 1}{\sin(\phi) - 1} \right| \right). \quad (2.13)$$

With regard to the integral borders, we keep that

$$\int_0^{\arctan(\frac{a_c}{y})} \frac{1}{\cos(\phi)} d\phi = \frac{1}{2} \cdot \ln \left( \left| \frac{\sin(\arctan(\frac{a_c}{y})) + 1}{\sin(\arctan(\frac{a_c}{y})) - 1} \right| \right), \quad (2.14)$$

whereas with  $\sin(\arctan(x)) = \frac{x}{\sqrt{x^2+1}}$  we can denote

$$\int_0^{\arctan(\frac{a_c}{y})} \frac{1}{\cos(\phi)} d\phi = \frac{1}{2} \cdot \ln \left( \left| \frac{\frac{\frac{a_c}{y}}{\sqrt{(\frac{a_c}{y})^2+1}} + 1}{\frac{\frac{a_c}{y}}{\sqrt{(\frac{a_c}{y})^2+1}} - 1} \right| \right) = \frac{1}{2} \cdot \ln \left( \left| \frac{\frac{1}{\sqrt{(\frac{y}{a_c})^2+1}} + 1}{\frac{1}{\sqrt{(\frac{y}{a_c})^2+1}} - 1} \right| \right). \quad (2.15)$$

Consequently, it follows that

$$\begin{aligned} \int_0^{\arctan(\frac{a_c}{y})} \frac{1}{\cos^3(\phi)} d\phi &= \frac{1}{2} \cdot \left[ \frac{1}{2} \cdot \ln \left( \left| \frac{\frac{1}{\sqrt{(\frac{y}{a_c})^2+1}} + 1}{\frac{1}{\sqrt{(\frac{y}{a_c})^2+1}} - 1} \right| \right) + \left( \frac{a_c}{y} \right)^2 \cdot \sqrt{\left( \frac{y}{a_c} \right)^2 + 1} \right] \\ &= \frac{1}{4} \cdot \ln \left( \left| \frac{\frac{1}{\sqrt{(\frac{y}{a_c})^2+1}} + 1}{\frac{1}{\sqrt{(\frac{y}{a_c})^2+1}} - 1} \right| \right) + \frac{1}{2} \cdot \left( \frac{a_c}{y} \right)^2 \cdot \sqrt{\left( \frac{y}{a_c} \right)^2 + 1} \\ &= \frac{1}{4} \cdot \ln \left( \left| \frac{\frac{a_c}{\sqrt{y^2+a_c^2}} + 1}{\frac{a_c}{\sqrt{y^2+a_c^2}} - 1} \right| \right) + \frac{1}{2} \cdot \frac{a_c}{y^2} \cdot \sqrt{y^2 + a_c^2}, \end{aligned} \quad (2.16)$$

and lastly

$$\int_0^{a_c} \sqrt{x^2 + y^2} dx = y^2 \cdot \left\{ \frac{1}{4} \cdot \ln \left( \left| \frac{\frac{a_c}{\sqrt{y^2 + a_c^2}} + 1}{\frac{a_c}{\sqrt{y^2 + a_c^2}} - 1} \right| \right) + \frac{1}{2} \cdot \frac{a_c}{y^2} \cdot \sqrt{y^2 + a_c^2} \right\}$$

$$\int_0^{a_c} \sqrt{x^2 + y^2} dx = \frac{y^2}{4} \cdot \ln \left( \left| \frac{\frac{a_c}{\sqrt{y^2 + a_c^2}} + 1}{\frac{a_c}{\sqrt{y^2 + a_c^2}} - 1} \right| \right) + \frac{1}{2} \cdot a_c \cdot \sqrt{y^2 + a_c^2}$$

(2.17)

### 3 Outer Integral

Having solved the inner integral, we have reduced the two dimensional problem to a one dimensional problem. Consequently, we continue with the outer integral, i. e.

$$\begin{aligned}
& \int_0^{a_c} \frac{y^2}{4} \cdot \ln \left( \left| \frac{\frac{a_c}{\sqrt{y^2+a_c^2}} + 1}{\frac{a_c}{\sqrt{y^2+a_c^2}} - 1} \right| \right) + \frac{1}{2} \cdot a_c \cdot \sqrt{y^2 + a_c^2} \, dy \\
&= \underbrace{\int_0^{a_c} \frac{y^2}{4} \cdot \ln \left( \left| \frac{a_c}{\sqrt{y^2+a_c^2}} + 1 \right| \right) \, dy}_{\text{Integral 'A'}} - \underbrace{\int_0^{a_c} \frac{y^2}{4} \cdot \ln \left( \left| \frac{a_c}{\sqrt{y^2+a_c^2}} - 1 \right| \right) \, dy}_{\text{Integral 'B'}} \dots \quad (3.1) \\
&\quad + \underbrace{\int_0^{a_c} \frac{1}{2} \cdot a_c \cdot \sqrt{y^2 + a_c^2} \, dy}_{\text{Integral 'C'}}.
\end{aligned}$$

Utilizing the linearity, we basically have to solve three independent integrals that we refer to as *Integral 'A'*, *Integral 'B'* and *Integral 'C'*. The solution for each of these is presented in the subsequent sections.

#### 3.1 Integral C

For *Integral 'C'*, we can adopt the result of the inner integral, so that

$$\begin{aligned}
& \int_0^{a_c} \frac{1}{2} \cdot a_c \cdot \sqrt{y^2 + a_c^2} \, dy = \frac{a_c}{2} \cdot \int_0^{a_c} \sqrt{y^2 + a_c^2} \, dy \\
&= \frac{a_c}{2} \cdot \left\{ \frac{a_c^2}{4} \cdot \ln \left( \left| \frac{\frac{a_c}{\sqrt{a_c^2+a_c^2}} + 1}{\frac{a_c}{\sqrt{a_c^2+a_c^2}} - 1} \right| \right) + \frac{1}{2} \cdot a_c \cdot \sqrt{a_c^2 + a_c^2} \right\} \quad (3.2) \\
&= \frac{a_c}{2} \cdot \left\{ \frac{a_c^2}{4} \cdot \ln \left( \left| \frac{\frac{1}{\sqrt{2}} + 1}{\frac{1}{\sqrt{2}} - 1} \right| \right) + \frac{1}{\sqrt{2}} \cdot a_c^2 \right\} = \frac{a_c^3}{8} \cdot \ln \left( \left| \frac{\frac{1}{\sqrt{2}} + 1}{\frac{1}{\sqrt{2}} - 1} \right| \right) + \frac{1}{2\sqrt{2}} \cdot a_c^3 \\
&= a_c^3 \cdot \left( \frac{1}{8} \cdot \ln \left( \left| \frac{1+\sqrt{2}}{1-\sqrt{2}} \right| \right) + \frac{1}{2\sqrt{2}} \right).
\end{aligned}$$

## 3.2 Integral A

We can rewrite *Integral 'A'* as

$$\int_0^{a_c} \frac{y^2}{4} \cdot \ln \left( \left| \frac{a_c}{\sqrt{y^2 + a_c^2}} + 1 \right| \right) dy = \frac{1}{4} \cdot \int_0^{a_c} y^2 \cdot \ln \left( \left| \frac{a_c}{\sqrt{y^2 + a_c^2}} + 1 \right| \right) dy, \quad (3.3)$$

and have to apply integration by parts, whereas

$$\begin{aligned} u(y) &= \frac{1}{3} \cdot y^3 \rightarrow \frac{du}{dy} = y^2 \quad \text{and} \quad v(y) = \ln \left( \left| \frac{a_c}{\sqrt{y^2 + a_c^2}} + 1 \right| \right) \\ \rightarrow \frac{dv}{dy} &= \frac{1}{\frac{a_c}{\sqrt{y^2 + a_c^2}} + 1} \cdot \frac{-\frac{1}{2} \cdot a_c}{(y^2 + a_c^2)^{\frac{3}{2}}} \cdot 2y = \frac{-a_c \cdot y}{(y^2 + a_c^2)^{\frac{3}{2}} + a_c \cdot y^2 + a_c^3}. \end{aligned} \quad (3.4)$$

Thus,

$$\begin{aligned} &\frac{1}{4} \cdot \int_0^{a_c} y^2 \cdot \ln \left( \left| \frac{a_c}{\sqrt{y^2 + a_c^2}} + 1 \right| \right) dy \\ &= \frac{1}{4} \cdot \left\{ \left[ \frac{1}{3} \cdot y^3 \cdot \ln \left( \left| \frac{a_c}{\sqrt{y^2 + a_c^2}} + 1 \right| \right) \right]_0^{a_c} - \int_0^{a_c} \frac{1}{3} \cdot y^3 \cdot \frac{-a_c \cdot y}{(y^2 + a_c^2)^{\frac{3}{2}} + a_c \cdot y^2 + a_c^3} dy \right\} \\ &= \frac{1}{12} \cdot a_c^3 \cdot \ln \left( \left| \frac{a_c}{\sqrt{a_c^2 + a_c^2}} + 1 \right| \right) - \frac{1}{12} \cdot \int_0^{a_c} y^3 \cdot \frac{-a_c \cdot y}{(y^2 + a_c^2)^{\frac{3}{2}} + a_c \cdot y^2 + a_c^3} dy \\ &= \frac{a_c^3}{12} \cdot \ln \left( \frac{1}{\sqrt{2}} + 1 \right) + \frac{a_c}{12} \cdot \int_0^{a_c} \frac{y^4}{(y^2 + a_c^2)^{\frac{3}{2}} + a_c \cdot y^2 + a_c^3} dy. \end{aligned} \quad (3.5)$$

Subsequently, we concentrate on

$$\int_0^{a_c} \frac{y^4}{(y^2 + a_c^2)^{\frac{3}{2}} + a_c \cdot y^2 + a_c^3} dy, \quad (3.6)$$

for which we perform a trigonometric substitution, where

$$\begin{aligned} y &= a_c \cdot \tan(\phi) \rightarrow \phi = \arctan \left( \frac{y}{a_c} \right) \\ \frac{dy}{d\phi} &= a_c \cdot \frac{1}{\cos^2(\phi)} \rightarrow dy = a_c \cdot \frac{1}{\cos^2(\phi)} d\phi, \end{aligned} \quad (3.7)$$

so, that

$$\begin{aligned}
 & \int \frac{y^4}{(y^2 + a_c^2)^{\frac{3}{2}} + a_c \cdot y^2 + a_c^3} dy \\
 &= \int \frac{(a_c \cdot \tan(\phi))^4}{((a_c \cdot \tan(\phi))^2 + a_c^2)^{\frac{3}{2}} + a_c \cdot (a_c \cdot \tan(\phi))^2 + a_c^3} \cdot a_c \cdot \frac{1}{\cos^2(\phi)} d\phi \\
 &= a_c^5 \cdot \int \frac{\tan(\phi)^4}{(a_c^2 \cdot \tan(\phi)^2 + a_c^2)^{\frac{3}{2}} + a_c^3 \cdot \tan(\phi)^2 + a_c^3} \cdot \frac{1}{\cos^2(\phi)} d\phi \\
 &= a_c^5 \cdot \int \frac{\tan(\phi)^4}{(a_c^2 \cdot (\tan(\phi)^2 + 1))^{\frac{3}{2}} + a_c^3 \cdot (\tan(\phi)^2 + 1)} \cdot \frac{1}{\cos^2(\phi)} d\phi \\
 &= a_c^5 \cdot \int \frac{\tan(\phi)^4}{a_c^3 \cdot (\tan(\phi)^2 + 1)^{\frac{3}{2}} + a_c^3 \cdot (\tan(\phi)^2 + 1)} \cdot \frac{1}{\cos^2(\phi)} d\phi \\
 &= a_c^2 \cdot \int \frac{\tan(\phi)^4}{(\tan(\phi)^2 + 1)^{\frac{3}{2}} + (\tan(\phi)^2 + 1)} \cdot \frac{1}{\cos^2(\phi)} d\phi,
 \end{aligned} \tag{3.8}$$

and with  $\tan(\phi)^2 + 1 = \frac{1}{\cos(\phi)^2}$ , we can further simplify the expression to

$$\begin{aligned}
 & \int \frac{y^4}{(y^2 + a_c^2)^{\frac{3}{2}} + a_c \cdot y^2 + a_c^3} dy \\
 &= a_c^2 \cdot \int \frac{\tan(\phi)^4}{(\tan(\phi)^2 + 1) \cdot ((\tan(\phi)^2 + 1)^{\frac{1}{2}} + 1)} \cdot (\tan(\phi)^2 + 1) d\phi \\
 &= a_c^2 \cdot \int \frac{\tan(\phi)^4}{(\tan(\phi)^2 + 1)^{\frac{1}{2}} + 1} d\phi = a_c^2 \cdot \int \frac{\tan(\phi)^4}{\sqrt{\tan(\phi)^2 + 1} + 1} d\phi \\
 &= a_c^2 \cdot \int \frac{\tan(\phi)^4}{\sqrt{\frac{1}{\cos(\phi)^2} + 1}} d\phi = a_c^2 \cdot \int \frac{\tan(\phi)^4 \cdot \cos(\phi)}{1 + \cos(\phi)} d\phi = a_c^2 \cdot \int \frac{\tan(\phi)^3 \cdot \sin(\phi)}{1 + \cos(\phi)} d\phi.
 \end{aligned} \tag{3.9}$$

Besides, using the identity  $\tan\left(\frac{x}{2}\right) = \frac{\sin(x)}{1+\cos(x)}$ , we can express the integral as

$$\begin{aligned}
 & \int \frac{y^4}{(y^2 + a_c^2)^{\frac{3}{2}} + a_c \cdot y^2 + a_c^3} dy \\
 &= a_c^2 \cdot \int \tan(\phi)^3 \cdot \frac{\sin(\phi)}{1 + \cos(\phi)} d\phi = a_c^2 \cdot \int \tan(\phi)^3 \cdot \tan\left(\frac{\phi}{2}\right) d\phi.
 \end{aligned} \tag{3.10}$$

Regarding the borders, we have to write

$$\int_0^{a_c} \frac{y^4}{(y^2 + a_c^2)^{\frac{3}{2}} + a_c \cdot y^2 + a_c^3} dy = a_c^2 \cdot \int_0^{\arctan\left(\frac{a_c}{a_c}\right) = \frac{\pi}{4}} \tan(\phi)^3 \cdot \tan\left(\frac{\phi}{2}\right) d\phi. \quad (3.11)$$

With  $\tan(x) = \frac{2 \cdot \tan\left(\frac{x}{2}\right)}{1 - \tan^2\left(\frac{x}{2}\right)}$ , we can further denote that

$$\begin{aligned} a_c^2 \cdot \int_0^{\frac{\pi}{4}} \tan(\phi)^3 \cdot \tan\left(\frac{\phi}{2}\right) d\phi &= a_c^2 \cdot \int_0^{\frac{\pi}{4}} \left( \frac{2 \cdot \tan\left(\frac{\phi}{2}\right)}{1 - \tan^2\left(\frac{\phi}{2}\right)} \right)^3 \cdot \tan\left(\frac{\phi}{2}\right) d\phi \\ &= a_c^2 \cdot \int_0^{\frac{\pi}{4}} \frac{8 \cdot \tan^4\left(\frac{\phi}{2}\right)}{1 - 3 \cdot \tan^2\left(\frac{\phi}{2}\right) + 3 \cdot \tan^4\left(\frac{\phi}{2}\right) - 1 \cdot \tan^6\left(\frac{\phi}{2}\right)} d\phi. \end{aligned} \quad (3.12)$$

Now, we can apply the Weierstraß substitution, i. e.  $x \tan\left(\frac{\phi}{2}\right) = t$  and  $d\phi = \frac{2 \cdot dt}{1+t^2}$ . So,

$$\begin{aligned} a_c^2 \cdot \int_0^{\frac{\pi}{4}} \tan(\phi)^3 \cdot \tan\left(\frac{\phi}{2}\right) d\phi &= a_c^2 \cdot \int_0^{\tan\left(\frac{\pi}{8}\right)} \frac{8t^4}{1 - 3t^2 + 3t^4 - t^6} \cdot \frac{2}{1+t^2} dt = 16 \cdot a_c^2 \cdot \int_0^{\tan\left(\frac{\pi}{8}\right)} \frac{t^4}{1 - 2t^2 + 2t^6 - t^8} dt. \end{aligned} \quad (3.13)$$

Using partial fraction decomposition, we can denote that

$$\begin{aligned} 16 \cdot a_c^2 \cdot \int_0^{\tan\left(\frac{\pi}{8}\right)} \frac{t^4}{1 - 2t^2 + 2t^6 - t^8} dt &= 16 \cdot a_c^2 \cdot \int_0^{\tan\left(\frac{\pi}{8}\right)} \frac{1}{32} \cdot \frac{1}{t-1} - \frac{3}{32} \cdot \frac{1}{(t-1)^2} - \frac{1}{16} \cdot \frac{1}{(t-1)^3} - \frac{1}{32} \cdot \frac{1}{t+1} \cdots \\ &\quad - \frac{3}{32} \cdot \frac{1}{(t+1)^2} + \frac{1}{16} \cdot \frac{1}{(t+1)^3} + \frac{1}{8} \cdot \frac{1}{t^2+1} dt \\ &= a_c^2 \cdot \int_0^{\tan\left(\frac{\pi}{8}\right)} \frac{1}{2} \cdot \frac{1}{t-1} - \frac{3}{2} \cdot \frac{1}{(t-1)^2} - \frac{1}{(t-1)^3} - \frac{1}{2} \cdot \frac{1}{t+1} \cdots \\ &\quad - \frac{3}{2} \cdot \frac{1}{(t+1)^2} + \frac{1}{(t+1)^3} + 2 \cdot \frac{1}{t^2+1} dt. \end{aligned} \quad (3.14)$$

Now, we have a series of standard integrals that we can solve. Accordingly,

$$16 \cdot a_c^2 \cdot \int_0^{\tan(\frac{\pi}{8})} \frac{t^4}{1 - 2t^2 + 2t^6 - t^8} dt = a_c^2 \cdot \left[ \frac{1}{2} \cdot \ln(|t - 1|) + \frac{3}{2} \cdot \frac{1}{t - 1} + \frac{1}{2(t - 1)^2} \cdots \right. \\ \left. - \frac{1}{2} \cdot \ln(|t + 1|) + \frac{3}{2} \cdot \frac{1}{t + 1} - \frac{1}{2(t + 1)^2} + 2 \cdot \arctan(t) \right]_0^{\tan(\frac{\pi}{8})}, \quad (3.15)$$

wherefore, we can give the solution of this integral as

$$\begin{aligned} & \int_0^{a_c} \frac{y^4}{(y^2 + a_c^2)^{\frac{3}{2}} + a_c \cdot y^2 + a_c^3} dy \\ &= a_c^2 \cdot \left[ \frac{1}{2} \cdot \ln \left( \left| \frac{\tan(\frac{\pi}{8}) - 1}{\tan(\frac{\pi}{8}) + 1} \right| \right) + \frac{3}{2} \cdot \left( \frac{1}{\tan(\frac{\pi}{8}) - 1} + \frac{1}{\tan(\frac{\pi}{8}) + 1} \right) \cdots \right. \\ & \quad \left. + \frac{1}{2} \cdot \left( \frac{1}{(\tan(\frac{\pi}{8}) - 1)^2} - \frac{1}{(\tan(\frac{\pi}{8}) + 1)^2} \right) + \frac{\pi}{4} \right] \\ &= \frac{a_c^2}{2} \cdot \left[ \ln \left( \left| \frac{\tan(\frac{\pi}{8}) - 1}{\tan(\frac{\pi}{8}) + 1} \right| \right) + 3 \cdot \left( \frac{1}{\tan(\frac{\pi}{8}) - 1} + \frac{1}{\tan(\frac{\pi}{8}) + 1} \right) \cdots \right. \\ & \quad \left. + \frac{1}{(\tan(\frac{\pi}{8}) - 1)^2} - \frac{1}{(\tan(\frac{\pi}{8}) + 1)^2} + \frac{\pi}{2} \right]. \end{aligned} \quad (3.16)$$

Thus, we can finish the calculation of the initial partial integral (see Equation 3.5) as

$$\begin{aligned} & \frac{1}{4} \cdot \int_0^{a_c} y^2 \cdot \ln \left( \left| \frac{a_c}{\sqrt{y^2 + a_c^2}} + 1 \right| \right) dy \\ &= \frac{a_c^3}{12} \cdot \ln \left( \frac{1}{\sqrt{2}} + 1 \right) + \frac{a_c^3}{24} \cdot \left[ \ln \left( \left| \frac{\tan(\frac{\pi}{8}) - 1}{\tan(\frac{\pi}{8}) + 1} \right| \right) + 3 \cdot \left( \frac{1}{\tan(\frac{\pi}{8}) - 1} + \frac{1}{\tan(\frac{\pi}{8}) + 1} \right) \cdots \right. \\ & \quad \left. + \frac{1}{(\tan(\frac{\pi}{8}) - 1)^2} - \frac{1}{(\tan(\frac{\pi}{8}) + 1)^2} + \frac{\pi}{2} \right] \\ &= \frac{a_c^3}{12} \cdot \left\{ \ln \left( \frac{1}{\sqrt{2}} + 1 \right) + \frac{1}{2} \cdot \left[ \ln \left( \left| \frac{\tan(\frac{\pi}{8}) - 1}{\tan(\frac{\pi}{8}) + 1} \right| \right) + 3 \cdot \left( \frac{1}{\tan(\frac{\pi}{8}) - 1} + \frac{1}{\tan(\frac{\pi}{8}) + 1} \right) \cdots \right. \right. \\ & \quad \left. \left. + \frac{1}{(\tan(\frac{\pi}{8}) - 1)^2} - \frac{1}{(\tan(\frac{\pi}{8}) + 1)^2} + \frac{\pi}{2} \right] \right\}. \end{aligned} \quad (3.17)$$

### 3.3 Integral B

Lastly, we solve *Integral 'B'* which is quite similar to *Integral 'A'* with the argument of the ln-function being the only slight difference. Consistently, we can start with

$$\int_0^{a_c} \frac{y^2}{4} \cdot \ln \left( \left| \frac{a_c}{\sqrt{y^2 + a_c^2}} - 1 \right| \right) dy = \frac{1}{4} \cdot \int_0^{a_c} y^2 \cdot \ln \left( \left| \frac{a_c}{\sqrt{y^2 + a_c^2}} - 1 \right| \right) dy, \quad (3.18)$$

and in turn apply integration by parts, whereas

$$\begin{aligned} u(y) &= \frac{1}{3} \cdot y^3 \rightarrow \frac{du}{dy} = y^2 \quad \text{and} \quad v(y) = \ln \left( \left| \frac{a_c}{\sqrt{y^2 + a_c^2}} - 1 \right| \right) \\ \rightarrow \frac{dv}{dy} &= \frac{1}{\frac{a_c}{\sqrt{y^2 + a_c^2}} - 1} \cdot \frac{-\frac{1}{2} \cdot a_c}{(y^2 + a_c^2)^{\frac{3}{2}}} \cdot 2y = \frac{a_c \cdot y}{(y^2 + a_c^2)^{\frac{3}{2}} - a_c \cdot y^2 - a_c^3}, \end{aligned} \quad (3.19)$$

so, that

$$\begin{aligned} \frac{1}{4} \cdot \int_0^{a_c} y^2 \cdot \ln \left( \left| \frac{a_c}{\sqrt{y^2 + a_c^2}} - 1 \right| \right) dy \\ &= \frac{1}{4} \cdot \left\{ \left[ \frac{1}{3} \cdot y^3 \cdot \ln \left( \left| \frac{a_c}{\sqrt{y^2 + a_c^2}} - 1 \right| \right) \right]_0^{a_c} - \int_0^{a_c} \frac{1}{3} \cdot y^3 \cdot \frac{a_c \cdot y}{(y^2 + a_c^2)^{\frac{3}{2}} - a_c \cdot y^2 - a_c^3} dy \right\} \\ &= \frac{a_c^3}{12} \cdot \ln \left( \left| \frac{1}{\sqrt{2}} - 1 \right| \right) - \frac{a_c}{12} \cdot \int_0^{a_c} \frac{y^4}{(y^2 + a_c^2)^{\frac{3}{2}} - a_c \cdot y^2 - a_c^3} dy. \end{aligned} \quad (3.20)$$

To solve

$$\int_0^{a_c} \frac{y^4}{(y^2 + a_c^2)^{\frac{3}{2}} - a_c \cdot y^2 - a_c^3} dy, \quad (3.21)$$

we perform a trigonometric substitution, where

$$\begin{aligned} y &= a_c \cdot \tan(\phi) \rightarrow \phi = \arctan \left( \frac{y}{a_c} \right) \\ \frac{dy}{d\phi} &= a_c \cdot \frac{1}{\cos^2(\phi)} \rightarrow dy = a_c \cdot \frac{1}{\cos^2(\phi)} d\phi. \end{aligned} \quad (3.22)$$

Resultantly, we can rewrite aforementioned integral to

$$\begin{aligned}
 & \int_0^{a_c} \frac{y^4}{(y^2 + a_c^2)^{\frac{3}{2}} - a_c \cdot y^2 - a_c^3} dy \\
 &= \int_0^{\frac{\pi}{4}} \frac{a_c^4 \cdot \tan^4(\phi)}{(a_c^2 \cdot \tan^2(\phi) + a_c^2)^{\frac{3}{2}} - a_c \cdot a_c^2 \cdot \tan^2(\phi) - a_c^3} \cdot a_c \cdot \frac{1}{\cos^2(\phi)} d\phi \\
 &= a_c^5 \cdot \int_0^{\frac{\pi}{4}} \frac{\tan^4(\phi)}{(a_c^2 \cdot (\tan^2(\phi) + 1))^{\frac{3}{2}} - a_c^3 \cdot (\tan^2(\phi) + 1)} \cdot \frac{1}{\cos^2(\phi)} d\phi \\
 &= a_c^2 \cdot \int_0^{\frac{\pi}{4}} \frac{\tan^4(\phi)}{(\tan^2(\phi) + 1)^{\frac{3}{2}} - (\tan^2(\phi) + 1)} \cdot \frac{1}{\cos^2(\phi)} d\phi,
 \end{aligned} \tag{3.23}$$

whereas with  $\tan(\phi)^2 + 1 = \frac{1}{\cos(\phi)^2}$  this expression becomes

$$\begin{aligned}
 & \int_0^{a_c} \frac{y^4}{(y^2 + a_c^2)^{\frac{3}{2}} - a_c \cdot y^2 - a_c^3} dy \\
 &= a_c^2 \cdot \int_0^{\frac{\pi}{4}} \frac{\tan^4(\phi)}{\left(\frac{1}{\cos(\phi)^2}\right)^{\frac{3}{2}} - \frac{1}{\cos(\phi)^2}} \cdot \frac{1}{\cos^2(\phi)} d\phi = a_c^2 \cdot \int_0^{\frac{\pi}{4}} \frac{\tan^4(\phi)}{\frac{1}{\cos(\phi)} - 1} d\phi \\
 &= a_c^2 \cdot \int_0^{\frac{\pi}{4}} \frac{\tan^4(\phi) \cdot \cos(\phi)}{1 - \cos(\phi)} d\phi = a_c^2 \cdot \int_0^{\frac{\pi}{4}} \frac{\tan^3(\phi) \cdot \sin(\phi)}{1 - \cos(\phi)} d\phi \\
 &= a_c^2 \cdot \int_0^{\frac{\pi}{4}} \tan^3(\phi) \cdot \frac{\sin(\phi)}{1 - \cos(\phi)} d\phi.
 \end{aligned} \tag{3.24}$$

Employing the identities  $\frac{1}{\tan(\frac{x}{2})} = \frac{\sin(x)}{1 - \cos(x)}$  and  $\tan(x) = \frac{2 \cdot \tan(\frac{x}{2})}{1 - \tan^2(\frac{x}{2})}$ , we can further denote that

$$\begin{aligned}
 & a_c^2 \cdot \int_0^{\frac{\pi}{4}} \left( \frac{2 \cdot \tan(\frac{\phi}{2})}{1 - \tan^2(\frac{\phi}{2})} \right)^3 \cdot \frac{1}{\tan(\frac{\phi}{2})} d\phi = a_c^2 \cdot \int_0^{\frac{\pi}{4}} \frac{8 \cdot \tan^2(\frac{\phi}{2})}{\left(1 - \tan^2(\frac{\phi}{2})\right)^3} d\phi \\
 &= 8 \cdot a_c^2 \cdot \int_0^{\frac{\pi}{4}} \frac{\tan^2(\frac{\phi}{2})}{1 - 3 \cdot \tan^2(\frac{\phi}{2}) + 3 \cdot \tan^4(\frac{\phi}{2}) - \tan^6(\frac{\phi}{2})} d\phi.
 \end{aligned} \tag{3.25}$$

Again, we rely on the Weierstraß substitution, i. e.  $\tan\left(\frac{\phi}{2}\right) = t$  and  $d\phi = \frac{2 \cdot dt}{1+t^2}$ . Hence,

$$\begin{aligned} & 8 \cdot a_c^2 \cdot \int_0^{\frac{\pi}{4}} \frac{\tan^2\left(\frac{\phi}{2}\right)}{1 - 3 \cdot \tan^2\left(\frac{\phi}{2}\right) + 3 \cdot \tan^4\left(\frac{\phi}{2}\right) - \tan^6\left(\frac{\phi}{2}\right)} d\phi \\ &= 8 \cdot a_c^2 \cdot \int_0^{\tan\left(\frac{\pi}{8}\right)} \frac{t^2}{1 - 3t^2 + 3t^4 - t^6} \cdot \frac{2}{1+t^2} dt = 16 \cdot a_c^2 \cdot \int_0^{\tan\left(\frac{\pi}{8}\right)} \frac{t^2}{1 - 2t^2 + 2t^6 - t^8} dt. \end{aligned} \quad (3.26)$$

Using partial fraction decomposition, this can be written as

$$\begin{aligned} 16 \cdot a_c^2 \cdot \int_0^{\tan\left(\frac{\pi}{8}\right)} \frac{t^2}{1 - 2t^2 + 2t^6 - t^8} dt &= a_c^2 \cdot \int_0^{\tan\left(\frac{\pi}{8}\right)} \frac{1}{2} \cdot \frac{1}{t-1} + \frac{1}{2} \cdot \frac{1}{(t-1)^2} - \frac{1}{(t-1)^3} \cdots \\ &\quad - \frac{1}{2} \cdot \frac{1}{t+1} + \frac{1}{2} \cdot \frac{1}{(t+1)^2} + \frac{1}{(t+1)^3} - 2 \cdot \frac{1}{t^2+1} dt. \end{aligned} \quad (3.27)$$

By that, we have rewritten the initial integral such, that we have the sum of standard integrals, which we can resolve as

$$\begin{aligned} & 16 \cdot a_c^2 \cdot \int_0^{\tan\left(\frac{\pi}{8}\right)} \frac{t^2}{1 - 2t^2 + 2t^6 - t^8} dt \\ &= a_c^2 \cdot \left[ \frac{1}{2} \cdot \ln(|t-1|) - \frac{1}{2} \cdot \frac{1}{t-1} + \frac{1}{2} \cdot \frac{1}{(t-1)^2} \cdots \right. \\ &\quad \left. - \frac{1}{2} \cdot \ln(|t+1|) - \frac{1}{2} \cdot \frac{1}{t+1} - \frac{1}{2} \cdot \frac{1}{(t+1)^2} - 2 \cdot \arctan(t) \right]_0^{\tan\left(\frac{\pi}{8}\right)} \\ &= a_c^2 \cdot \left[ \frac{1}{2} \cdot \ln\left(\left|\frac{\tan\left(\frac{\pi}{8}\right) - 1}{\tan\left(\frac{\pi}{8}\right) + 1}\right|\right) - \frac{1}{2} \cdot \left(\frac{1}{\tan\left(\frac{\pi}{8}\right) - 1} + \frac{1}{\tan\left(\frac{\pi}{8}\right) + 1}\right) \cdots \right. \\ &\quad \left. + \frac{1}{2} \cdot \left(\frac{1}{\left(\tan\left(\frac{\pi}{8}\right) - 1\right)^2} - \frac{1}{\left(\tan\left(\frac{\pi}{8}\right) + 1\right)^2}\right) - \frac{\pi}{4} \right] \\ &= \frac{a_c^2}{2} \cdot \left[ \ln\left(\left|\frac{\tan\left(\frac{\pi}{8}\right) - 1}{\tan\left(\frac{\pi}{8}\right) + 1}\right|\right) - \frac{1}{\tan\left(\frac{\pi}{8}\right) - 1} - \frac{1}{\tan\left(\frac{\pi}{8}\right) + 1} \cdots \right. \\ &\quad \left. + \frac{1}{\left(\tan\left(\frac{\pi}{8}\right) - 1\right)^2} - \frac{1}{\left(\tan\left(\frac{\pi}{8}\right) + 1\right)^2} - \frac{\pi}{2} \right]. \end{aligned} \quad (3.28)$$

Thus, the solution for *Integral 'B'* can be given as

$$\begin{aligned}
 & \frac{1}{4} \cdot \int_0^{a_c} y^2 \cdot \ln \left( \left| \frac{a_c}{\sqrt{y^2 + a_c^2}} - 1 \right| \right) dy \\
 &= \frac{a_c^3}{12} \cdot \ln \left( \left| \frac{1}{\sqrt{2}} - 1 \right| \right) - \frac{a_c}{12} \cdot \int_0^{a_c} \frac{y^4}{(y^2 + a_c^2)^{\frac{3}{2}} - a_c \cdot y^2 - a_c^3} dy \\
 &= \frac{a_c^3}{12} \cdot \left\{ \ln \left( \left| \frac{1}{\sqrt{2}} - 1 \right| \right) - \frac{1}{2} \cdot \left[ \ln \left( \left| \frac{\tan(\frac{\pi}{8}) - 1}{\tan(\frac{\pi}{8}) + 1} \right| \right) - \frac{1}{\tan(\frac{\pi}{8}) - 1} - \frac{1}{\tan(\frac{\pi}{8}) + 1} \right. \right. \\
 &\quad \left. \left. + \frac{1}{(\tan(\frac{\pi}{8}) - 1)^2} - \frac{1}{(\tan(\frac{\pi}{8}) + 1)^2} - \frac{\pi}{2} \right] \right\}.
 \end{aligned} \tag{3.29}$$

With that, we have finally solved all integrals. Accordingly, we can give the overall analytical solution for the initial problem in the next chapter.

## 4 Result

With the preliminary work presented in the foregoing sections, we are finally able to give the analytical solution for the initial integral as

$$\begin{aligned}
\mu_d &= \frac{1}{a_c^2} \cdot \left\{ a_c^3 \cdot \left( \frac{1}{8} \cdot \ln \left( \left| \frac{1+\sqrt{2}}{1-\sqrt{2}} \right| \right) + \frac{1}{2\sqrt{2}} \right) \dots \right. \\
&\quad + \frac{a_c^3}{12} \cdot \left\{ \ln \left( \frac{1}{\sqrt{2}} + 1 \right) + \frac{1}{2} \cdot \left[ \ln \left( \left| \frac{\tan(\frac{\pi}{8}) - 1}{\tan(\frac{\pi}{8}) + 1} \right| \right) + 3 \cdot \left( \frac{1}{\tan(\frac{\pi}{8}) - 1} + \frac{1}{\tan(\frac{\pi}{8}) + 1} \right) \dots \right. \right. \\
&\quad \left. \left. + \frac{1}{(\tan(\frac{\pi}{8}) - 1)^2} - \frac{1}{(\tan(\frac{\pi}{8}) + 1)^2} + \frac{\pi}{2} \right] \right\} \dots \\
&\quad - \frac{a_c^3}{12} \cdot \left\{ \ln \left( \left| \frac{1}{\sqrt{2}} - 1 \right| \right) - \frac{1}{2} \cdot \left[ \ln \left( \left| \frac{\tan(\frac{\pi}{8}) - 1}{\tan(\frac{\pi}{8}) + 1} \right| \right) - \frac{1}{\tan(\frac{\pi}{8}) - 1} - \frac{1}{\tan(\frac{\pi}{8}) + 1} \dots \right. \right. \\
&\quad \left. \left. + \frac{1}{(\tan(\frac{\pi}{8}) - 1)^2} - \frac{1}{(\tan(\frac{\pi}{8}) + 1)^2} - \frac{\pi}{2} \right] \right\} \right\} \\
&= a_c \cdot \left\{ \frac{1}{8} \cdot \ln \left( \left| \frac{1+\sqrt{2}}{1-\sqrt{2}} \right| \right) + \frac{1}{2\sqrt{2}} + \frac{1}{12} \cdot \left[ \dots \right. \right. \\
&\quad \ln \left( \frac{1}{\sqrt{2}} + 1 \right) + \frac{1}{2} \cdot \ln \left( \left| \frac{\tan(\frac{\pi}{8}) - 1}{\tan(\frac{\pi}{8}) + 1} \right| \right) + \frac{3}{2} \cdot \left( \frac{1}{\tan(\frac{\pi}{8}) - 1} + \frac{1}{\tan(\frac{\pi}{8}) + 1} \right) \dots \\
&\quad + \frac{1}{2} \cdot \frac{1}{(\tan(\frac{\pi}{8}) - 1)^2} - \frac{1}{2} \cdot \frac{1}{(\tan(\frac{\pi}{8}) + 1)^2} + \frac{\pi}{4} \dots \\
&\quad - \ln \left( \left| \frac{1}{\sqrt{2}} - 1 \right| \right) + \frac{1}{2} \cdot \ln \left( \left| \frac{\tan(\frac{\pi}{8}) - 1}{\tan(\frac{\pi}{8}) + 1} \right| \right) - \frac{1}{2} \cdot \frac{1}{\tan(\frac{\pi}{8}) - 1} - \frac{1}{2} \cdot \frac{1}{\tan(\frac{\pi}{8}) + 1} \dots \\
&\quad \left. \left. + \frac{1}{2} \cdot \frac{1}{(\tan(\frac{\pi}{8}) - 1)^2} - \frac{1}{2} \cdot \frac{1}{(\tan(\frac{\pi}{8}) + 1)^2} - \frac{\pi}{4} \right] \right\} \tag{4.1}
\end{aligned}$$

$$\begin{aligned}
 \mu_d &= a_c \cdot \left\{ \frac{1}{8} \cdot \ln \left( \left| \frac{1+\sqrt{2}}{1-\sqrt{2}} \right| \right) + \frac{1}{2\sqrt{2}} + \frac{1}{12} \cdot \left[ \ln \left( \frac{1}{\sqrt{2}} + 1 \right) - \ln \left( \left| \frac{1}{\sqrt{2}} - 1 \right| \right) \dots \right. \right. \\
 &\quad + \frac{1}{2} \cdot \ln \left( \left| \frac{\tan(\frac{\pi}{8}) - 1}{\tan(\frac{\pi}{8}) + 1} \right| \right) + \frac{1}{2} \cdot \ln \left( \left| \frac{\tan(\frac{\pi}{8}) - 1}{\tan(\frac{\pi}{8}) + 1} \right| \right) \dots \\
 &\quad + \frac{3}{2} \cdot \frac{1}{\tan(\frac{\pi}{8}) - 1} - \frac{1}{2} \cdot \frac{1}{\tan(\frac{\pi}{8}) - 1} + \frac{3}{2} \cdot \frac{1}{\tan(\frac{\pi}{8}) + 1} - \frac{1}{2} \cdot \frac{1}{\tan(\frac{\pi}{8}) + 1} \dots \\
 &\quad + \frac{1}{2} \cdot \frac{1}{(\tan(\frac{\pi}{8}) - 1)^2} + \frac{1}{2} \cdot \frac{1}{(\tan(\frac{\pi}{8}) - 1)^2} \dots \\
 &\quad \left. \left. - \frac{1}{2} \cdot \frac{1}{(\tan(\frac{\pi}{8}) + 1)^2} - \frac{1}{2} \cdot \frac{1}{(\tan(\frac{\pi}{8}) + 1)^2} + \frac{\pi}{4} - \frac{\pi}{4} \right] \right\} \\
 &= a_c \cdot \left\{ \frac{1}{8} \cdot \ln \left( \left| \frac{1+\sqrt{2}}{1-\sqrt{2}} \right| \right) + \frac{1}{2\sqrt{2}} + \frac{1}{12} \cdot \left[ \ln \left( \left| \frac{1+\sqrt{2}}{1-\sqrt{2}} \right| \right) + \ln \left( \left| \frac{\tan(\frac{\pi}{8}) - 1}{\tan(\frac{\pi}{8}) + 1} \right| \right) \dots \right. \right. \\
 &\quad + \frac{1}{\tan(\frac{\pi}{8}) - 1} + \frac{1}{\tan(\frac{\pi}{8}) + 1} + \frac{1}{(\tan(\frac{\pi}{8}) - 1)^2} - \frac{1}{(\tan(\frac{\pi}{8}) + 1)^2} \left. \right] \right\} \\
 &= a_c \cdot \left\{ \frac{5}{24} \cdot \ln \left( \left| \frac{1+\sqrt{2}}{1-\sqrt{2}} \right| \right) + \frac{1}{2\sqrt{2}} + \frac{1}{12} \cdot \left[ \ln \left( \left| \frac{\tan(\frac{\pi}{8}) - 1}{\tan(\frac{\pi}{8}) + 1} \right| \right) \dots \right. \right. \\
 &\quad + \frac{1}{\tan(\frac{\pi}{8}) - 1} + \frac{1}{\tan(\frac{\pi}{8}) + 1} + \frac{1}{(\tan(\frac{\pi}{8}) - 1)^2} - \frac{1}{(\tan(\frac{\pi}{8}) + 1)^2} \left. \right] \right\}. \tag{4.2}
 \end{aligned}$$

With

$$\tan\left(\frac{\pi}{8}\right) = \tan\left(\frac{1}{2} \cdot \frac{\pi}{4}\right) = \frac{\sin\left(\frac{\pi}{4}\right)}{1 + \cos\left(\frac{\pi}{4}\right)} = \frac{\frac{1}{\sqrt{2}}}{1 + \frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2} + 1}, \tag{4.3}$$

or, respectively,

$$\tan\left(\frac{\pi}{8}\right) = \tan\left(\frac{1}{2} \cdot \frac{\pi}{4}\right) = \frac{1 - \cos\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)} = \frac{1 - \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \sqrt{2} - 1, \tag{4.4}$$

we can rewrite the expression to

$$\begin{aligned}
 \mu_d &= a_c \cdot \left\{ \frac{5}{24} \cdot \ln \left( \left| \frac{1+\sqrt{2}}{1-\sqrt{2}} \right| \right) + \frac{1}{2\sqrt{2}} + \frac{1}{12} \cdot \left[ \ln \left( \left| \frac{\sqrt{2}-1-1}{\sqrt{2}-1+1} \right| \right) \dots \right. \right. \\
 &\quad \left. \left. + \frac{1}{\sqrt{2}-1-1} + \frac{1}{\sqrt{2}-1+1} + \frac{1}{(\sqrt{2}-1-1)^2} - \frac{1}{(\sqrt{2}-1+1)^2} \right] \right\} \\
 &= a_c \cdot \left\{ \frac{5}{24} \cdot \ln \left( \left| \frac{1+\sqrt{2}}{1-\sqrt{2}} \right| \right) + \frac{1}{12} \cdot \ln (|1-\sqrt{2}|) \dots \right. \\
 &\quad \left. + \frac{1}{2\sqrt{2}} + \frac{1}{12} \cdot \left[ \frac{1}{\sqrt{2}-2} + \frac{1}{\sqrt{2}} + \frac{1}{6-4\sqrt{2}} - \frac{1}{2} \right] \right\} \\
 &= a_c \cdot \left\{ \frac{5}{24} \cdot \ln \left( \left| \frac{1+\sqrt{2}}{1-\sqrt{2}} \right| \right) + \frac{1}{12} \cdot \ln (|1-\sqrt{2}|) + \frac{1}{2\sqrt{2}} + \frac{7-5\sqrt{2}}{42\sqrt{2}-60} \right\} \\
 &= a_c \cdot \left\{ \frac{5}{24} \cdot \ln \left( \left| \frac{1+\sqrt{2}}{1-\sqrt{2}} \right| \right) + \frac{1}{12} \cdot \ln (|1-\sqrt{2}|) + \frac{1}{3} \cdot \frac{7\sqrt{2}-10}{7-5\sqrt{2}} \right\} \\
 &= a_c \cdot \left\{ \frac{5}{12} \cdot \ln (|1+\sqrt{2}|) + \frac{1}{12} \cdot \ln (|1-\sqrt{2}|) + \frac{1}{3} \cdot \frac{7\sqrt{2}-10}{7-5\sqrt{2}} \right\} \\
 &= a_c \cdot \frac{1}{3} \cdot \left\{ \frac{5}{4} \cdot \ln (|1+\sqrt{2}|) + \frac{1}{4} \cdot \ln (|1-\sqrt{2}|) + \frac{7\sqrt{2}-10}{7-5\sqrt{2}} \right\} \\
 &= a_c \cdot \frac{1}{3} \cdot \left\{ \frac{5}{4} \cdot \ln (|1+\sqrt{2}|) + \frac{1}{4} \cdot \ln (|1-\sqrt{2}|) + \sqrt{2} \right\} \\
 &= a_c \cdot \frac{1}{3} \cdot \left\{ \ln (|1+\sqrt{2}|) + \frac{1}{4} \cdot \ln (|1+\sqrt{2}|) + \frac{1}{4} \cdot \ln (|1-\sqrt{2}|) + \sqrt{2} \right\} \\
 &= a_c \cdot \frac{1}{3} \cdot \left\{ \ln (|1+\sqrt{2}|) + \frac{1}{4} \cdot \ln (|(1+\sqrt{2})(1-\sqrt{2})|) + \sqrt{2} \right\} \\
 \boxed{\mu_d = a_c \cdot \frac{1}{3} \cdot [\ln (1+\sqrt{2}) + \sqrt{2}]}
 \end{aligned} \tag{4.5}$$